RIEMANNIAN MANIFOLDS WITHOUT CONJUGATE POINTS: A LECTURE ON A THEOREM OF HOPF AND GREEN

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1. Introduction

In 1948 E. Hopf [3] proved that any Riemannian metric on the two dimensional torus that is without conjugate points is a flat metric. The proof proceeds by showing that any metric on a compact surface without conjugate has non-positive Gaussian curvature and then using the Gauss-Bonnet theorem to conclude that when the surface is a torus that the Gaussian curvature is identically zero. In 1958 L. Green generalized Hopf's argument to show that any metric on a compact Riemannian manifold of any dimension that is without conjugate points has non-positive scalar curvature. The note here is based on Green's paper gives an elementary exposition of the Hopf-Green result, however the proof is just a reworking of Green's proof with no changes of substance.

2. Systems of Ordinary Differential Equations without Conjugate Points

In this section $t \mapsto R(t)$ will be a smooth map form the real numbers $\mathbf{R}$ into the vector space of $m \times m$ symmetric matrices. For any $a \in \mathbf{R}$ let $S(t;a)$ be defined by the initial value problem

$$S''(t;a) + R(t)S(t;a) = 0, \quad S(a;a) = 0, \quad S'(a,a) = I \tag{2.1}$$

where I is the $m \times m$ identity matrix. We say that $R(t)$ is ***free of conjugate points*** iff of all $a \in \mathbf{R}$ and $t \neq a$ we have $\det S(t;a) \neq 0$. If $R(t)$ is free of conjugate points, then for $t \neq a$ define

$$A(t;a) = -S'(t,a)S(t,a)^{-1}.$$

If we view (2.1) as the Jacobi equations along a geodesic $\gamma(t)$ in a Riemannian manifold then the condition that $R(t)$ is free of conjugate points is exactly that the geodesic is free of conjugate points in the usual sense. If $t < a$ then $A(t;a)$ is the second fundamental form (viewed as a $(1,1)$ tenser) of the geodesic sphere centered at $\gamma(t)$ and passing through $\gamma(t)$ with respect

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to the normal d/dt. The horosphere determined by this geodesic is objected obtained by taking the geometric limit of these geodesic spheres as $a \to \infty$. The following result more or less says that these second fundamental form of these horospheres exits and that it satisfies the correct matrix Riccati equation.

Theorem 2.1 (E. Hopf [3] ($m = 1$) and L. Green [2] ($m \geq 2$))**.** *If $R(t)$ is free of conjugate points then*

$$U(t) := \lim_{a \to \infty} A(t; a)$$

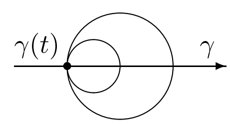
exists for all t, the function $t \mapsto U(t)$ is smooth and satisfies the Riccati equation

$$U'(t) = U(t)^2 + R(t).$$

If A and B are symmetric $m \times m$ matrices then $A \leq B$ means that $B - A$ is positive semi-definite. Likewise $A < B$ will mean that $B - A$ is positive definite.

Lemma 2.2. *Under the hypothesis of the theorem, if $t < a < b$, then $A(t; a) > A(t; b)$.*

Proof. We first note that if $A(t, a)$ and $A(t; b)$ are the second fundamental forms of the geodesic spheres centered at $\gamma(a)$ and $\gamma(b)$ and through $\gamma(t)$ then the triangle inequality implies the geodesic sphere centered at $\gamma(b)$.



This can be translated into the desired inequality. (If two hypersurfaces are tangent at a point and one lies on one side of the other, then there is an inequality between the second fundamental forms.) To give an analytic proof we first note that a direct calculation shows that $A(t; a)$ satisfies a Riccati equation $A'(t; a) = A(t; a)^2 + R(t)$. Also from the initial value problem defining $S(t, a)$ near $t = a$

$$\begin{aligned} S(t; a) &= (t - a)I + O(t - a)^3, \\ S(t; a)^{-1} &= \frac{1}{(t - a)} I + O(t - a)^3, \\ S'(t; a) &= I + O(t - a)^2. \end{aligned}$$

Thus

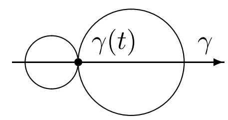
$$A(t; a) = -S'(t; a)S(t; a)^{-1} = \frac{-1}{(t - a)} I + O(t - a). \tag{2.2}$$

For t just a little smaller than a we thus see that $A(t; a)$ is of approximately of the form CI for C large and positive. As $a < b$ this implies $A(t, a) > A(t, b)$ for t just a little smaller than a. But then the comparison theory for the Riccati equation [1, Sec. 3] implies $A(t, a) > A(t, b)$ for all $t < a$. $\square$

Lemma 2.3. *With the hypothesis of the theorem, if $a, b > 0$ then $A(t; a) > A(t; b)$ for $-b < t < a$.*

Proof. Geometrically if $\gamma(t)$ is a line, that is if it minimizes the distance between any two of its points, and $A(t,a)$ and $A(t;b)$ are the second fundamental forms of the geodesic spheres centered at $\gamma(a)$ and $\gamma(b)$ and through $\gamma(t)$ then the triangle inequality implies the geodesic sphere centered at $\gamma(a)$ is outside the geodesic sphere centered at $\gamma(b)$. As in the last lemma this implies an inequality between the second fundamental forms.

Analytically we again use equation (2.2). If a is replaced by $-b$ in (2.2) then for t just a little larger than $-b$ we see that $A(t,-a)$ is approximately $-CI$ for C a large positive constant. Thus for t just a little larger than $-a$ we have $A(t;-a) < A(t;b)$ and thus the comparison theory implies $A(t;-a) < A(t;b)$ for $-a < t < b$. □



Proof of Theorem 2.1. Fix $c > 0$ and let $a > c$. Lemma 2.3 implies that on the interval $[-c,c]$ we have $A(t,a) > A(t,-2c)$. Thus for some constant (only depending on $R(t)$ and c) there holds $-CI \leq A(t,a)$ for all $t \in [-c,c]$ and $a > c$. By Lemma 2.2 there for fixed $t \in [-c,c]$ $A(t,a)$ is a decreasing function of a. As there is a lower bound, we see that $U(t) = \lim_{a\to\infty} A(t;a)$ exists for all $t \in [-c,c]$. For $t \in [-c,c]$, $a > 2c$ we see that $A'(t;a) = A(t;a)^2 + R(t)$ stays bounded, so $A(t;a)$ is uniformly Lipschitz and thus by Ascoli's theorem the convergence in the limit is uniform. This in turn implies that as $a \to \infty$ that $A'(t;a) = A(t;a)^2 + R(t)$ converges uniformly to something, and it is easy to see that this something must be $U'(t)$. Thus $U'(t) = U(t)^2 + R(t)$. As $U(t)$ satisfies a an ordinary differential equation it is a smooth function. □

3. Riemannian Manifolds without Conjugate Points

Let (M,g) be a compact n dimensional Riemannian manifold without conjugate without conjugate points. Let $S(M)$ be the unit sphere bundle of M and let ζ^t be the geodesic flow on $S(M)$. Then, as usual, the geodesic flow preserves the natural volume measure on $S(M)$. For each $u \in S(M)$ let $\gamma_u(t)$ be the geodesic fitting u (that is $\gamma_u'(0) = u$). Then as γ is without conjugate points we can construct a field linear maps $U(t)$ along γ so that for each t $U(t)$ is a selfadjoint linear map $\gamma_u'(t)^\perp$ that satisfies $U'(t) = U(t)^2 + R(t)$ where $R(t)$ is defined by $R(t)X := R(X, \gamma_u'(t))\gamma_u'(t)$ and $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ is the curvature tensor of (M,g). Define a function U on $S(M)$ by $U_u := U(0)$. Thus if M is simply connected U_u is the second fundamental form of the horosphere determined by γ_u through the base point of u. For $u \in S(M)$ the function $t \mapsto U_{\zeta^t u}$ is smooth, but I do not know if the dependence of U_u on u is continuous, but I assume that it is not. However the map $u \mapsto U_u$ is measurable (in [2] Green refers us to Hopf's paper [3] which I have yet to look at. But as far as I am concerned all functions that come up in geometry problems are measurable.)

We now take traces of the differential equation $U' = U^2 + R$ and use the invariance of the Liouville measure under the geodesic flow.

$$\begin{aligned}
0 &= \int_{S(M)} \operatorname{tr}(U_{\zeta^1 u} - U_u)\, du && \text{invariance of } du \\
&= \int_{S(M)} \int_0^1 \operatorname{tr}(U'_{\zeta^t u})\, dt\, du \\
&= \int_{S(M)} \int_0^1 \operatorname{tr}(U^2_{\zeta^t u} + R_{\zeta^t u})\, dt\, du \\
&= \int_{S(M)} \operatorname{tr}(U_u^2) + \operatorname{tr}(R_u)\, du && \text{invariance of } du \\
&= \int_{S(M)} \operatorname{tr}(U_u^2) + \operatorname{Ric}(u,u)\, du && \text{definition of Ric} \\
&= \int_{S(M)} \operatorname{tr}(U_u^2)\, du + \frac{\operatorname{Vol} S^{n-1}}{n} \int_M \operatorname{Scal}\, dx
\end{aligned}$$

where at the last step we have used that $\int_{S^{n-1}} \operatorname{Ric}(u,u)\, du = \frac{\operatorname{Vol} S^{n-1}}{n} \operatorname{Scal}$ where Scal is the scalar curvature of (M, g). This gives the formula

$$\int_M \operatorname{Scal}\, dx = -\frac{n}{\operatorname{Vol} S^{n-1}} \int_{S(M)} \operatorname{tr}(U_u^2)\, du.$$

This implies at once that if (M, g) is compact without conjugate points and then the integral of Scal is non-positive and if $\int_M \operatorname{Scal}\, dx = 0$ then $U_u = 0$ almost everywhere. This implies $R_u = U_u' - U_u^2 = 0$ for almost all $u \in S(M)$. Therefore M must be flat. When $n = 2$ this is due to E. Hopf [3] and for $n \geq 3$ it is due to L. Green [2]. Note that if $n = 2$ and M is a torus then by the Gauss-Bonnet theorem $\int_M \operatorname{Scal}\, dx = 0$. Thus a metric without conjugate points on a torus is flat.

References

1. L. Andersson and R. Howard, *Comparison and rigidity theorems in semi-Riemannian geometry*, Communications in Analysis and Geometry **6** (1998), no. 4, 819–877.
2. L. Green, *A theorem of E. Hopf*, Michigan Math. Jour. **5** (1958), 31–34.
3. E. Hopf, *Closed surfaces without conjugate points*, Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 47–51.