A FUNCTIONAL EQUATION CHARACTERIZATION OF ARCHIMEDEAN ORDERED FIELDS

RALPH HOWARD, VIRGINIA JOHNSON, AND GEORGE F. MCNULTY

Abstract. We prove that an ordered field is Archimedean if and only if every continuous additive function from the field to itself is linear over the field.

In 1821 Cauchy, [1], observed that any continuous function S on the real line that satisfies

$$S(x+y) = S(x) + S(y) \quad \text{for all reals } x \text{ and } y$$

is just multiplication by a constant. Another way to say this is that S is a linear operator on $\mathbb{R}$, viewing $\mathbb{R}$ as a vector space over itself. The constant is evidently $S(1)$. The displayed equation is *Cauchy's functional equation* and solutions to this equation are called *additive*. To see that Cauchy's result holds, note that only a small amount of work is needed to verify the following steps: first $S(0) = 0$, second $S(-x) = -S(x)$, third $S(nx) = S(x)n$ for all integers, and finally that $S(r) = S(1)r$ for every rational number r. But then S and the function $x \mapsto S(1)x$ are continuous functions that agree on a dense set (the rationals) and therefore are equal. So Cauchy's result follows, in part, from the fact that the rationals are dense in the reals. In 1875 Darboux, in [2], extended Cauchy's result by noting that if an additive function is continuous at just one point, then it is continuous everywhere. Therefore the conclusion of Cauchy's theorem holds under the weaker hypothesis that S is just continuous at a single point. In 1905 Hamel, in [5], showed Cauchy's result was nontrivial by proving the existence of discontinuous additive functions $f\colon \mathbb{R} \to \mathbb{R}$.

The field of real numbers is an ordered field. In general, a field $\mathbb{F}$ is *ordered* by a 2-place relation $\leq$ provided each of the following properties holds:

- the relation $\leq$ is a linear ordering of $\mathbb{F}$,
- for all x, y, z, and w in $\mathbb{F}$, if $x \leq y$ and $z \leq w$, then $x + z \leq y + w$, and
- for all x, y, and z in $\mathbb{F}$, if $x \leq y$ and $0 \leq z$, then $xz \leq yz$.

Every ordered field has characteristic 0, allowing us to regard the field of rational numbers, $\mathbb{Q}$, as a subfield.

If $\mathbb{F}$ is a field ordered by $\leq$, then, exactly analogous to the real numbers, it has the *order topology* (which has as a base the open intervals $(a, b) := \{x \in F : a < x < b\}$). A function $f\colon \mathbb{F} \to \mathbb{F}$ is continuous with respect to this topology at $a \in \mathbb{F}$ if and only if the usual ε-δ definition holds. That is, for all $\varepsilon \in \mathbb{F}$ with $\varepsilon > 0$, there is a $\delta \in \mathbb{F}$ with $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$. (Here $|x| = \max\{x, -x\}$ is the absolute value defined by the order on $\mathbb{F}$.)

An ordered field $\mathbb{F}$ is *Archimedean* provided any one (and hence all) of the following logically equivalent conditions holds:

(a) For every x in $\mathbb{F}$, if $0 < x$, then there is a positive integer n so that $0 < 1/n < x$.

(b) The set of rationals is dense in $\mathbb{F}$.
(c) The ordered field $\mathbb{F}$ is embeddable into the ordered field of real numbers.

(Proofs of the equivalence of the equivalence of the third condition with the first two be found in [4, Theorem 3.5] or [9, Section 4].)

Observe that the failure of condition (a) asserts the existence of some nonzero element ε of $\mathbb{F}$ so that $0 \leq |\varepsilon| < 1/n$ for every positive integer n. An element ε that satisfies those inequalities is called an *infinitesimal*. So 0 is infinitesimal and the failure of (a) asserts the existence of a nonzero infinitesimal. There are non-Archimedean ordered fields. Perhaps the most familiar is the field $\mathbb{R}(x)$ of rational functions with real coefficients. This field is ordered by the relation $\sqsubseteq$ of eventual domination. That is, given rational functions f and g we put

$$f \sqsubseteq g \text{ if and only if } f(x) \leq g(x) \text{ for all large enough values of } x.$$

The rational function $1/x$ is an infinitesimal in this ordered field. Actually, the class of ordered fields is rich in such examples and a large part of the theory of ordered fields concerns those that are not Archimedean.

The geometric form of the Archimedean property is the Axiom of Archimedes: Given two segments there is a positive integer, n, such that the longer segment can be divided into n equal pieces of length less than the shorter segment. Archimedes attributes this to Eudoxus of Cnidus who used it to justify the method of exhaustion. The first examples of non-Archimedean geometries and fields were given by David Hilbert in his book on the foundations of geometry [6] in 1899.

Recently there has been renewed interest in Archimedean fields and their characterizations. The survey paper [3] of Deveau and Teismann contains 42 statements about an ordered field that are equivalent to the field being Archimedean (and 72 statements equivalent to it being complete). In [9] Propp considers the Axiom of Archimedes in the context of “reverse mathematics.” Other recent results about Archimedean fields are given in [4, 8, 10, 11] and our preprint [7] where a generalization of the result given here is proved.

Here we show that Cauchy's result is also equivalent to the Archimedean property. First a bit of notation. An ordered field $\mathbb{F}$ is also a vector space over the rational numbers, denoted by $\mathbb{F}_{\mathbb{Q}}$, and a vector space over itself, denoted by $\mathbb{F}_{\mathbb{F}}$.

Theorem. *Let $\mathbb{F}$ be an ordered field with its order topology. Then $\mathbb{F}$ is Archimedean if and only if every continuous function on $\mathbb{F}$ that satisfies Cauchy's functional equation is a linear operator on $\mathbb{F}_{\mathbb{F}}$.*

Proof. In the case that $\mathbb{F}$ is Archimedean, the rational numbers are dense in $\mathbb{F}$ and the argument given above works just as in the case when $\mathbb{F} = \mathbb{R}$.

For the converse, let $\mathbb{F}$ be a non-Archimedean ordered field. We will construct a continuous additive function, T, on $\mathbb{F}$ that is not linear on $\mathbb{F}_{\mathbb{F}}$. We will even show that T can be taken to be bijective. It is somewhat ironic that the idea of the proof is based on the method Hamel used in [5] to show that there are discontinuous solutions to Cauchy's equation on $\mathbb{R}$.

Let $\mathbb{I}$ be the set of infinitesimals in $\mathbb{F}$, that is, the set of $x \in \mathbb{F}$ such that $|x| < 1/n$ for all natural numbers n. As $\mathbb{F}$ is non-Archimedean, the set $\mathbb{I} \neq \{0\}$. Both $\mathbb{F}_{\mathbb{Q}}$ and $\mathbb{I}$ are vector spaces over $\mathbb{Q}$. Let $\mathcal{B}_{1/\infty}$ be a basis for $\mathbb{I}$. Since 1 is not infinitesimal, $\mathcal{B}_{1/\infty} \cup \{1\}$ is linearly independent. (If 1 were linearly dependent on elements of $\mathbb{I}$ with coefficients from the rationals, then it would be an infinitesimal.) Extend $\mathcal{B}_{1/\infty} \cup \{1\}$ to a basis $\mathcal{B}$ of $\mathbb{F}$ over $\mathbb{Q}$.

For all $a \in \mathbb{F}$, define $T(a)$ to be the coefficient of 1 in the representation of a as a linear combination of elements of $\mathcal{B}$. So T is a linear operator on $\mathbb{F}_{\mathbb{Q}}$. From the definition, $T(b) = b$ for all $b \in \mathbb{Q}$ and $T(\varepsilon) = 0$ for all infinitesimals ε.

Since T is a linear map on $\mathbb{F}_{\mathbb{Q}}$, it satisfies Cauchy's functional equation. We need to show that T is not linear on $\mathbb{F}_{\mathbb{F}}$. If T were linear on $\mathbb{F}_{\mathbb{F}}$, then for any nonzero $\varepsilon \in \mathbb{I}$, one would have $0 = T(\varepsilon) = \varepsilon T(1) = \varepsilon$, a contradiction.

It remains to show that the map T is continuous. As noted before, $T(u) = 0$ for all $u \in \mathbb{I}$. Because $\mathbb{I} \neq \{0\}$, there is a $\delta \in \mathbb{I}$ with $\delta > 0$. Let $a \in \mathbb{F}$ and let ε be any positive element of $\mathbb{F}$. If $x \in \mathbb{F}$ and $|x - a| < \delta$, then $x - a \in \mathbb{I}$ and therefore $T(x - a) = 0$. Thus by additivity,

$$|x - a| < \delta \text{ implies } |T(x) - T(a)| = |T(x - a)| = 0 < \varepsilon.$$

Whence T is not only continuous, but uniformly continuous. $\square$

This map T constructed in the proof is not injective. To see that there is a bijective example, let $a, b \in \mathbb{Q}$ and define $T_{a,b} \colon \mathbb{F} \to \mathbb{F}$ as

$$T_{a,b}(x) = aT(x) + bx \text{ for all } x \in \mathbb{F}.$$

Each $T_{a,b}$ is continuous and satisfies Cauchy's functional equation. Recalling that T is a linear map over $\mathbb{Q}$ and using that $a, b \in \mathbb{Q}$, it is easy to see that $T_{c,d}$ is the inverse of $T_{a,b}$, where $c = -a/(b(a + b))$ and $d = 1/b$, provided $b(a + b) \neq 0$. So, for example, $T_{1,1}(x) = T(x) + x$ is invertible and its inverse is $T_{-\frac{1}{2},1}(x) = -\frac{1}{2}T(x) + x$. But if $a \neq 0$, then $T_{a,b}$ is not linear on $\mathbb{F}_{\mathbb{F}}$, for if it were, then

$$T = \frac{1}{a}(T_{a,b} - bI)$$

would be linear on $\mathbb{F}_{\mathbb{F}}$ and we have already seen this is not the case.

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References

[1] A. L. B. Cauchy, *Cours d'analyse de l'école royale polytechnique: Analyse algébrique*, Debure frères, 1821.

[2] J. G. Darboux, Mémoire sur les fonctions discontinues. (french), *Ann. Sci. cole Norm. Sup.* **2** (1875) 57–112.

[3] M. Deveau, H. Teismann, $72 + 42$: Characterizations of the completeness and Archimedean properties of ordered fields, *Real Anal. Exchange* **39** (2013/14) no. 2, 261–303.

[4] J. F. Hall, Completeness of Ordered Fields, ArXiv e-prints (2011). `https://arxiv.org/abs/1101.5652`

[5] G. Hamel, Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung: $f(x + y) = f(x) + f(y)$, *Math. Ann.* **60** (1905) no. 3, 459–462.

[6] D. Hilbert, *The Foundations of Geometry.* Authorized translation by E. J. Townsend. Reprint edition, The Open Court Publishing Co., La Salle, Ill., 1959.

[7] R. Howard, V. Johnson, G. F. McNulty, Maps that must be affine or conjugate affine: A problem of Vladimir Arnold, Preprint, 2016.

[8] R. Kantrowitz, M. Neumann, Another face of the archimedean property, *College Math. J.* **46** (2015) no. 2, 139–141.

[9] J. Propp, Real analysis in reverse, *Amer. Math. Monthly* **120** (2013), no. 5, 392–408.i

[10] O. Riemenschneider, 37 elementare axiomatische Charakterisierungen des reellen Zahlkörpers, *Mitt. Math. Ges. Hamburg* **20** (2001), 71–95.

[11] H. Teismann, Toward a more complete list of completeness axioms, *Amer. Math. Monthly* **120** (2013), no. 2, 99–114.

Department of Mathematics, University of South Carolina, Columbia SC 29208
E-mail address: howard@math.sc.edu

Department of Mathematics, University of South Carolina, Columbia SC 29208
E-mail address: vjohnson@columbiasc.edu

Department of Mathematics, University of South Carolina, Columbia SC 29208
E-mail address: mcnulty@math.sc.edu