A GENERAL THEORY OF ALMOST CONVEX FUNCTIONS.

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Abstract. Let $\Delta_m = \{(t_0, \ldots, t_m) \in \mathbf{R}^{m+1} : t_i \geq 0, \sum_{i=0}^m t_i = 1\}$ be the standard m-dimensional simplex. Let $\varnothing \neq S \subset \bigcup_{m=1}^\infty \Delta_m$, then a function $h\colon C \to \mathbf{R}$ with domain a convex set in a real vector space is ***S-almost convex*** iff for all $(t_0, \ldots, t_m) \in S$ and $x_0, \ldots, x_m \in C$ the inequality

$$h(t_0x_0 + \cdots + t_mx_m) \leq 1 + t_0h(x_0) + \cdots + t_mh(x_m)$$

holds. A detailed study of the properties of S-almost convex functions is made. If S contains at least one point that is not a vertex, then an extremal S-almost convex function $E_S\colon \Delta_n \to \mathbf{R}$ is constructed with the properties that it vanishes on the vertices of Δ_n and if $h\colon \Delta_n \to \mathbf{R}$ is any bounded S-almost convex function with $h(e_k) \leq 0$ on the vertices of Δ_n, then $h(x) \leq E_S(x)$ for all $x \in \Delta_n$. In the special case $S = \{(1/(m+1), \ldots, 1/(m+1))\}$, the barycenter of Δ_m, very explicit formulas are given for E_S and $\kappa_S(n) = \sup_{x \in \Delta_n} E_S(x)$. These are of interest as E_S and $\kappa_S(n)$ are extremal in various geometric and analytic inequalities and theorems.

Contents

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1. Introduction.

Let C be a convex set in a real vector space and let $h\colon C \to R$. Then according to Hyers and Ulam [5] for $\varepsilon > 0$, h is ε-approximately convex iff

$$h((1-t)x + ty) \le \varepsilon + (1-t)h(x) + th(y), \quad \text{for all} \quad t \in [0,1]. \tag{1.1}$$

In [5] they show that if h is ε-approximately convex and $C \subseteq \mathbf{R}^n$ then there is a convex function $g\colon C \to \mathbf{R}$ and a constant $C(n)$ only depending on the dimension so that $|h(x) - g(x)| \le \frac{1}{2}C(n)\varepsilon$. In a previous paper we show the sharp constant is

$$C(n) = \lfloor \log_2 n \rfloor + \frac{2(n+1-2^{\lfloor \log_2 n \rfloor})}{n+1}.$$

(Here $\lfloor \cdot \rfloor$ is the floor, or greatest integer function, and $\lceil \cdot \rceil$ is the ceiling function, that is $\lceil x \rceil$ is the smallest integer greater than or equal to x.) In the present paper we generalize the notion of approximate convexity and give the sharp constants in the corresponding Hyers-Ulam type theorems. This is done by finding the extremal approximately convex function on the simplex that vanishes on the vertices.

Let us put the these problems in a somewhat larger setting. First, by replacing h by $\varepsilon^{-1}h$ in (1.1), there is no loss of generality in assuming that $\varepsilon = 1$. Then many natural notions of generalized convexity are covered in the following definition. Let $\Delta_m = \{(t_0, \dots, t_m) \in \mathbf{R}^{m+1} : t_i \ge 0, \sum_{i=0}^m t_i = 1\}$ be the standard m-dimensional simplex.

1.1. **Definition.** Let V a vector space over the reals and let $\varnothing \ne C \subseteq V$ be a convex set and let $\varnothing \ne S \subseteq \bigcup_{m=1}^\infty \Delta_m$. Then a function $h\colon C \to \mathbf{R}$ is S-***almost convex*** on C iff for all $(t_0, \dots, t_m) \in S$ and $x_0, \dots, x_m \in C$ the inequality

$$h\left(\sum_{i=0}^m t_i x_i\right) \le 1 + \sum_{i=0}^m t_i h(x_i)$$

holds. We denote by

$$\mathrm{AlmCon}_S(C) := \{h : h \text{ is } S\text{-almost convex on } C\}$$

the set of almost convex functions $h\colon C \to \mathbf{R}$. □

The case of $S = \Delta_1$ corresponds to the case studied by Hyers and Ulam [5] and others (cf. the book [4] for more information and references). When $S = \{(1/2, 1/2)\}$ the S-almost convex functions are just the functions that satisfy

$$h\left(\frac{x+y}{2}\right) \leq 1 + \frac{h(x)+h(y)}{2}.$$

which are the approximately midpoint convex functions, (sometimes called the approximately Jensen convex functions) which also have been studied by several authors.

We give a general theory of S-almost convex functions. In particular when S has at least one point that is not a vertex we construct (Definition 1.17 and Theorem 1.22) a bounded S-almost convex function $E_S^{\Delta_n}\colon \Delta_n \to \mathbf{R}$ such that if $h\colon \Delta_n \to \mathbf{R}$ is bounded, S-almost convex, and $h(e_k) \leq 0$ on the vertices of Δ_n then $h(x) \leq E_S^{\Delta_n}(x)$ for all $x \in \Delta_n$. Then the number $\kappa_S(n) := \sup_{x\in\Delta_n} E_S^{\Delta_n}(x)$ is the sharp constant in stability theorems of Hypers-Ulam type and the function $E_S^{\Delta_m}$ is the function that shows it is sharp (See Theorem 1.26.)

Probably the most natural choices, for S are $S = \Delta_m$, a simplex, and $S = \{(1/(m+1), \dots, 1/(m+1))\}$, the barycenter of a simplex. In these cases we are able to give very explicit formulas both for the extremal function $E_S^{\Delta_n}$ and for the constant $\kappa_S(n) = \sup_{x\in\Delta_n} E_S^{\Delta_n}(x)$. (For the case $S = \Delta_m$ this was done in our earlier paper [3] where

$$\kappa_{\Delta_m}(n) = \lfloor \log_{m+1} n \rfloor + \frac{\left\lceil (m+1)\left((n+1) - (m+1)^{\lfloor \log_{m+1} n \rfloor}\right)/m \right\rceil}{n+1}.$$

For the case of S the barycenter of Δ_m see Theorem 3.1, where the value is given as

$$\kappa_{\{(1/(m+1),\dots,1/(m+1))\}}(n) = \lfloor \log_{m+1} n \rfloor + 1 + \frac{n}{m(m+1)^{\lfloor \log_{m+1} n \rfloor}}. \tag{1.2}$$

(This differs from the notation of Theorem 3.1 by the substitution $B = m+1$.)) There is an interesting dichotomy in these two cases. When $S = \Delta_m$ then $E_S^{\Delta_n}$ is a concave piecewise linear function that is continuous on the interior Δ_n° of Δ_n and the maximum occurs at the barycenter of Δ_n. (See [3].) However when $S = \{(1/(m+1), \dots, 1/(m+1))\}$ is the barycenter of Δ_m then $E_S^{\Delta_n}$ is discontinuous on a dense subset of Δ_n and the graph of $E_S^{\Delta_n}$ is a fractal with a large number of self similarities and the maximum does *not* occur at the barycenter of Δ_m. See Figure 2. We also note the somewhat surprising fact, that, as functions of n, both $\kappa_{\Delta_m}(n)$ and $\kappa_{\{(1/(m+1),\ldots,1/(m+1))\}}(n)$ have the same order of growth, i.e. $\lfloor \log_{m+1} n \rfloor + O(1)$.

This paper is not completely self-contained. Several of the results have proofs that are very similar to the proofs in our earlier paper [2] and at several places we refer the reader to [2] for proofs.

1.1. **Definition and basic properties.** Let $\Delta_m := \{(t_0, \ldots, t_m) : \sum_{k=0}^m t_k = 1, t_k \geq 0\}$ be the standard m-dimensional simplex. For the rest of this section we fix a subset

$$S \subset \bigcup_{m=1}^{\infty} \Delta_m.$$

It follows easily from the definition of S-almost convex that $\mathrm{AlmCon}_S(C)$ is a convex subset of the vector space of all functions from C to $\mathbf{R}$.

It is useful to make a distinction between two cases:

1.2**. Definition.** If $S \subseteq \bigcup_{m=1}^{\infty} \Delta_m$ then

(1) If $S \not\subseteq \bigcup_{m=1}^{N} \Delta_m$ for any finite N then S is of ***infinite type***.
(2) If $S \subseteq \bigcup_{m=1}^{N} \Delta_m$ for some N then S is of ***finite type***. If further $S \subseteq \Delta_m$ for some m then S is ***homogeneous***. □

1.3. *Remark.* If we assume that the union $\bigcup_{m=1}^{\infty} \Delta_m$ is disjoint and has the natural topology ($U \subseteq \bigcup_{m=1}^{\infty} \Delta_m$ is open iff $U \cap \Delta_m$ is open in Δ_m for all m) then it is not hard to see that S is of finite type if and only if it has compact closure in $\bigcup_{m=1}^{\infty} \Delta_m$. □

When considering S-almost convex functions there is no real distinction between S of finite type and S homogeneous.

1.4**. Proposition.** *Let* $S \subseteq \bigcup_{m=1}^{N} \Delta_m$*. For* $m \leq N$ *let* $\iota_N^m \colon \Delta_m \to \Delta_N$ *be the inclusion* $\iota_N^n(t_0, \ldots, t_m) = (t_0, \ldots, t_m, 0, \ldots, 0)$ *and set* $S_m^* = \iota_N^m[S \cap \Delta_m] \subseteq \Delta_N$*. Let* $S^* = \bigcup_{m=1}^{N} S_m^* \subseteq \Delta_N$*. Then for any convex subset* C *of a real vector space* $\mathrm{AlmCon}_{S^*}(C) = \mathrm{AlmCon}_S(C)$*.*

Proof. This is a more or less straightforward chase though the definition. □

The proof of the following is also straightforward and left to the reader.

1.5**. Proposition.** *Let* $S \subseteq \Delta_m$ *and let*

$$S^* = \bigcup_{\rho \in \mathrm{sym}(m+1)} \{(t_{\rho(0)}, t_{\rho(1)}, \ldots, t_{\rho(m)}) : (t_0, t_1, \ldots, t_m) \in S\}$$

where $\operatorname{sym}(m+1)$ *is the group of all permutations of* $\{0,1,\ldots,m\}$. *Then for any convex subset* C *of a real vector space* $\operatorname{AlmCon}_{S^*}(C) = \operatorname{AlmCon}_S(C)$.

The following is also trivial.

1.6. **Proposition.** *Let* $S_1 \subseteq S_2 \subseteq \bigcup_{m=1}^\infty \Delta_m$. *Then for any convex subset* C *of a real vector space* $\operatorname{AlmCon}_{S_2}(C) \subseteq \operatorname{AlmCon}_{S_1}(C)$. □

The following can be used to reduce certain questions about S-almost convex functions to the case where $S \subseteq \Delta_1$.

1.7. **Proposition.** *Let* $S \subseteq \bigcup_{m=1}^\infty \Delta_m$ *and let* S_1 *be a nonempty subset of* $S \cap \Delta_m$ *for some* m. *Let* $N_0,\ldots,N_k$ *be a partition of the set* $\{0,1,\ldots,m\}$ *into* $k+1$ *nonempty sets and let*

$$S_2 := \{(\alpha_0(t), \alpha_2(t), \ldots, \alpha_k(t)) : t \in S_1\} \subseteq \Delta_k$$

where

$$\alpha_j(t) := \sum_{i \in N_j} t_i.$$

Then

$$\operatorname{AlmCon}_S(C) \subseteq \operatorname{AlmCon}_{S_2}(C)$$

for any convex subset C *of a real vector space. In particular if* $(t_0,\ldots,t_m) \in S$ *and for some* $k \in \{0,\ldots,m-1\}$ *we set* $\alpha = t_0 + \cdots + t_k$ *and* $\beta = t_{k+1} + \cdots + t_m$ *then any* S *almost convex function* h *will satisfy* $h(\alpha x_0) + h(\beta x_1) \le 1 + \alpha h(x_0) + \beta h(x_1)$.

Proof. Let C be a convex subset of a real vector space and let $y_0,\ldots,y_k \in C$, $\alpha \in S_2$ and $h \in \operatorname{AlmCon}_S(C)$. Let $x_0,\ldots,x_m \in C$ be defined by

$$x_i = y_j \quad \text{if} \quad i \in N_j$$

As $\alpha \in S_2$ there is a $t = (t_0,\ldots,t_m) \in S_1 \subseteq S$ so that $\alpha_j = \sum_{i \in N_j} t_i$. Then as h is S-almost convex

$$h\left(\sum_{j=0}^k \alpha_j y_j\right) = h\left(\sum_{i=0}^m t_i x_i\right) \le 1 + \sum_{i=0}^k t_i h(x_i) = 1 + \sum_{i=0}^m \alpha_j h(y_j).$$

Thus $h \in \operatorname{AlmCon}_{S_2}(C)$. □

It is useful to understand when an S-almost convex function is bounded.

1.8. **Theorem.** *Let* $S \subseteq \bigcup_{m=1}^\infty \Delta_m$ *and assume that* S *contains at least one point that is not a vertex (that is there is* $(t_0,\ldots,t_m) \in S$ *with* $\max_i t_i < 1$*). Let* U *be a convex open set in* $\mathbf{R}^n$. *Then any* S*-almost convex function* $h\colon U \to \mathbf{R}$ *which is Lebesgue measurable is bounded above and below on any compact subset of* U.

Proof. Let $(t_0, \dots, t_m) \in S$ with $\max_i t_i < 1$. Then there is a $k \in \{0, \dots, m-1\}$ so that if $\alpha = t_0 + \cdots + t_k$ and $\beta = t_{k+1} + \cdots + t_m$, then $0 < \alpha, \beta < 1$, $\alpha + \beta = 1$ and by Proposition 1.7

$$h(\alpha x_0 + \beta x_1) \leq 1 + \alpha h(x_0) + \beta h(x_1).$$

We assume that $\alpha \leq \beta$, the case of $\alpha > \beta$ having a similar proof. As any compact subset of U is contained in a bounded convex open subset of U we can also assume, without loss of generality, that U is bounded.

Let $K \subset U$ be compact and let $r = \operatorname{dist}(K, \partial U)$. For any $x \in \mathbf{R}^n$ let $B_r(x)$ be the open ball of radius r about x. Then for any $a \in K$ we have $B_r(a) \subseteq U$. For $a \in K$ define $\theta_a \colon \mathbf{R}^n \to \mathbf{R}^n$ by

$$\theta_a(x) = \frac{1}{\beta} a - \frac{\alpha}{\beta} x.$$

Then it is easy to check that $\theta_a(a) = a$ for all $a \in \mathbf{R}^n$ and $\alpha x + \beta \theta_a(x) = a$ for all $x \in \mathbf{R}^n$. Also θ_a is a dilation in the sense that $\|\theta_a(x_1) - \theta_a(x_0)\| = (\alpha/\beta)\|x_1 - x_0\|$ for all $x_0, x_1 \in \mathbf{R}^n$. As $\theta_a(a) = a$ and $(\alpha/\beta) \leq 1$ this implies $\theta_a[B_a(r)] = B_a((\alpha/\beta)r) \subseteq B_a(r)$. Let $\mathcal{L}^n$ be Lebesgue measure on $\mathbf{R}^n$. Then for any measurable subset P of $\mathbf{R}^n$

$$\mathcal{L}^n(\theta_a[P]) = (\alpha/\beta)^n \mathcal{L}^n(P).$$

Choose a positive real number ε so that

$$\left(1 + \left(\frac{\alpha}{\beta}\right)^n\right)\varepsilon < \left(\frac{\alpha}{\beta}\right)^n \mathcal{L}^n(B(r)) \tag{1.3}$$

where $B(r)$ is the open ball of radius r about the origin. Because h is measurable and $\mathcal{L}^n(U) < \infty$ there is a positive M so large that

$$\mathcal{L}^n\{x \in U : h(x) > M\} < \varepsilon.$$

Therefore if $V := \{x \in U : h(x) \leq M\}$ then $\mathcal{L}^n(U \setminus V) < \varepsilon$. Let $A := B_a(r) \cap V$. We now claim that $A \cap \theta_a[A]$ has positive measure. For if not then A and $\theta_a[A]$ would be essentially disjoint subsets of $B_r(a)$ and therefore, using that $\mathcal{L}^n(\theta_a[A]) = (\alpha/\beta)^n \mathcal{L}^n(A)$,

$$\begin{aligned}
\mathcal{L}^n(B_a(r)) &\geq \mathcal{L}^n(A) + \mathcal{L}^n(\theta_a[A]) \\
&= \left(1 + \left(\frac{\alpha}{\beta}\right)^n\right) \mathcal{L}^n(A) \\
&\geq \left(1 + \left(\frac{\alpha}{\beta}\right)^n\right) (\mathcal{L}^n(B_a(r)) - \varepsilon)
\end{aligned}$$

which can be rearranged as $(1 + (\alpha/\beta)^n)\,\varepsilon \geq (\alpha/\beta)^n \mathcal{L}^n(B(r))$ contradicting (1.3). Therefore $\mathcal{L}^n(A \cap \theta_a[A]) > 0$ as claimed. Let $a \neq x \in$

$A \cap \theta_a[A]$). Then x and $\theta_a(x)$ are both in $A = B_a(r) \cap V$ and therefore $h(x), h(\theta_a(x)) \le M$. Thus

$$h(a) = h(\alpha x + \beta\theta_a(x)) \le 1 + \alpha h(x) + \beta h(\theta_a(x)) \le 1 + \alpha M + \beta M = M + 1$$

which shows that h is bounded above on K.

To show that h has a lower bound on compact subsets of U, let $a \in U$ and let $r > 0$ be small enough that the closed ball $\overline{B}_a(r)$ is contained in U. Then $\overline{B}_a(r)$is compact so by what we have just done there is a constant $C > 0$ so that $h(x) \le C$ for all $x \in B_a(r)$. Let $x \in B_a(r)$. Then, again as above, $\theta_a(x) \in B_a(r)$, and therefore

$$h(a) = h(\alpha x + \beta\theta_a(x)) \le 1 + \alpha h(x) + \beta h(\theta_a(x)) \le 1 + \alpha h(x) + \beta C$$

which can be solved for $h(x)$ to give

$$h(x) \ge \frac{1}{\alpha}(h(a) - 1 - \beta C).$$

Therefore h is bounded below on $B_a(r)$. But any compact subset of U can be covered by a finite number of such open balls and thus h is bounded below on all compact subsets of U. □

The following will be needed later.

1.9**. Corollary.** *Let $h\colon [a, b] \to \mathbf{R}$ be a Lebesgue measurable function so that $h(\alpha x + \beta y) \le 1 + \alpha h(x) + \beta h(y)$ for some $\alpha, \beta > 0$ with $\alpha + \beta = 1$ (that is h is S-almost convex with $S = \{(\alpha, \beta)\} \subset \Delta_1$). Then h is bounded above on $[a, b]$.*

Proof. By doing a linear change of variable (which preserves S-almost convexity) we can assume that $[a, b] = [0, 1]$. Also by replacing h by $x \mapsto h(x) - ((1 - x)h(0) + xh(1))$ we can assume that $h(0) = h(1) = 0$. Let $\delta = \alpha/(1 + \alpha)$. Then by Theorem 1.8 there is a constant $C_1 > 0$ such that $h(x) \le C_1$ on $[\delta, 1 - \delta]$. Let

$$C_2 = \max\{C_1, 1/(1 - \alpha) + \alpha C_1\}.$$

We now show that $h \le C_2$ on $[0, 1]$. If $x = 0$, $x = 1$, or $x \in [\delta, 1 - \delta]$ this is clear. Let $x \in (0, \delta)$ then the choice of δ ensures that there is a $y \in [\delta, 1 - \delta]$ such that $x = \alpha^k y$ for some positive integer k. Also, as $y \in [\delta, 1 - \delta]$, $h(y) \le C_1$. Therefore

$$\begin{aligned} h(x) &= h(\alpha^k y) = h(\beta 0 + \alpha\alpha^{k-1} y) \\ &\le 1 + \beta h(0) + \alpha h(\alpha^{k-1} y) = 1 + \alpha h(\alpha^{k-1} y) \\ &\le 1 + \alpha\big(1 + \alpha h(\alpha^{k-2} y)\big) = 1 + \alpha + \alpha^2 h(\alpha^{k-2} y) \\ &\le 1 + \alpha + \alpha^2 + \cdots + \alpha^{k-1} + \alpha^k h(y) \end{aligned}$$

$$\leq \frac{1}{1-\alpha} + \alpha C_1 \leq C_2.$$

If $x \in (1-\delta, 1)$ a similar calculation shows that $h(x) \leq C_2$ (or this can be reduced to the case $x \in (0, \delta)$ by the change of variable $x \mapsto (1-x)$). This completes the proof. □

1.2. **A general construction for the extremal S almost convex function on a simplex.** We will show that on the n-dimensional simplex Δ_n there is a pointwise largest bounded S-almost convex function that vanishes on the vertices of Δ_m. We start with some definitions.

1.10**. Definition.** A ***tree***, T, is a collection of points $\mathcal{N}$, called ***nodes***, and a set of (directed) ***edges*** connecting some pairs of nodes with the following properties: The set $\mathcal{N}$ is a disjoint union $\mathcal{N} = \bigcup_{k=0}^{\infty} \mathcal{N}_k$ where $\mathcal{N}_0$ contains exactly one point, the ***root of the tree***, each $\mathcal{N}_k$ is a finite set and if $\mathcal{N}_k = \{v_1, \ldots, v_m\}$ then $\mathcal{N}_{m+1}$ is a disjoint union $\mathcal{N}_{m+1} = \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_m$ of nonempty sets where $\mathcal{P}_i$ is the set of ***successors of*** v_i. The (directed) edges of the tree leave a node and connect it to its successors and there are no other edges in the tree (cf. Figure 1). If v is a node of the tree then $r(v) := k$ where $v \in \mathcal{N}_k$ is the ***rank*** of v. A ***branch*** of the tree is a sequence of nodes $\langle v_k \rangle_{k=0}^{\infty}$ where v_0 is the root, $r(v_k) = k$, and there is an edge from v_k to v_{k+1}. □

We now consider trees with extra structure, a labeling of the edges in a way that will be used in defining the extremal S-almost convex function.

1.11**. Definition.** Let $S \subseteq \bigcup_{m=1}^{\infty} \Delta_m$ be nonempty. Then an ***S-ranked tree*** is a tree T with its edges labeled by non-negative real numbers in such a way that for any node v of the tree there is an element $t = (t_0, \ldots, t_m) \in S$ so that there are exactly $m+1$ edges leaving v and these are labeled by $t_0, \ldots, t_m$. The number t_i is the ***weight*** of the edge it labels. Figure 1 shows a typical S-ranked tree. □

We now describe how an S-ranked tree determines a probability measure on the set of branches of the tree. Let T be an S-ranked tree and let $X = X(T)$ be the set of all branches of T. If $\langle v_k \rangle_{k=0}^{\infty}, \langle w_k \rangle_{k=0}^{\infty} \in X$ are two elements of X we can define a distance between them as $d(\langle v_k \rangle_{k=0}^{\infty}, \langle w_k \rangle_{k=0}^{\infty}) = 2^{-\ell}$ where ℓ is the smallest index with $v_\ell \neq w_\ell$ (and $d(\langle v_k \rangle_{k=0}^{\infty}, \langle w_k \rangle_{k=0}^{\infty}) = 0$ if $\langle v_k \rangle_{k=0}^{\infty} = \langle w_k \rangle_{k=0}^{\infty}$). While we will not need to use this fact, it is not hard to check that this makes X into a compact metric space which is homeomorphic to the Cantor set.

1.12**. Definition.** Let $S \subseteq \bigcup_{m=1}^{\infty} \Delta_m$ be nonempty and let T be an S-ranked tree. Then T defines a measure on X, the set of branches of

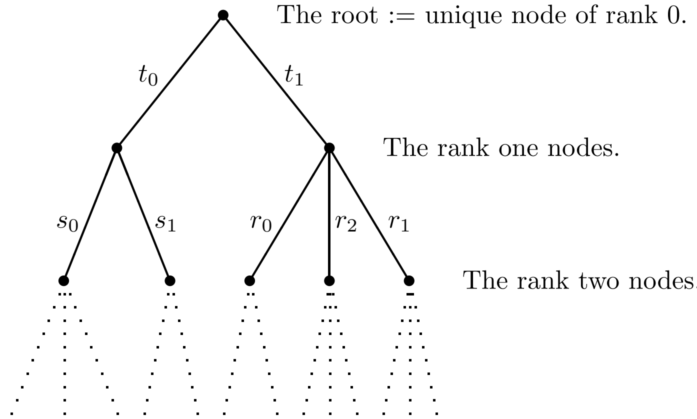

FIGURE 1. An S ranked tree showing the labeling of the edges out of the root by $t = (t_0, t_1) \in S$ and the edges out of the rank one nodes by $s = (s_0, s_1) \in S$ and $r = (r_0, r_1, r_2) \in S$. In our definition each node will have at least two edges leaving it and the sum of the weights $t_0, \dots, t_m$ of the weights of all edges leaving a node is unity (as $(t_0, \dots, t_m) \in \Delta_m$). Finally, in the definition of tree used here, all branches are of infinite length.

T, as follows. For v a node of T let $I(v)$ be the set of branches of T that pass through v. If $k = r(v)$ is the rank of v then let $\langle v_0, v_1, \dots, v_k \rangle$ be the initial segment of a branch passing through v (so that $v = v_k$) and for $1 \leq i \leq k$ let s_i be the weight of the edge from v_{i-1} to v_i. Then μ is the measure on X such that

$$\mu(I(v)) = s_0 s_1 \cdots s_k.$$

(That is $\mu(I(v))$ is the product of the weights of the edges along an initial segment of a branch connecting the root to v.) A measure arising in this way will be called an ***S-ranked probability measure***. □

It follows from this definition that if v is a node of T and $v_0, \dots, v_m$ are the successors of v and $t = (t_0, \dots, t_m) \in S$ labels the edges from v in such a way that t_i labels the edge from v to v_i then

$$\mu(I(v_i)) = t_i \mu(I(v)).$$

It is useful to give a description of an S-ranked probability measure that does not rely directly on its construction from an S-ranked tree.

1.13. **Alternative Definition.** An ***S-ranked probability measure*** is an ordered triple $(X, \mu, \boldsymbol{\pi})$ where X is a nonempty set, $\boldsymbol{\pi} = \langle \pi_0, \pi_1, \pi_2, \dots \rangle$ a sequence of finite partitions of X into nonempty subsets such that $\pi_0 = \{X\}$ and π_{k+1} refines π_k, μ is a measure defined on the σ-algebra, $\mathcal{A}(\boldsymbol{\pi})$, generated by $\bigcup_{k=0}^{\infty} \pi_k$ so that for all $j \geq 0$ and all $I \in \pi_j$, there exists $(t_0, \ldots, t_m) \in S$ such that if

$$\{J \in \pi_{j+1} : J \subset I\} = \{I_0, I_1, \ldots, I_m\}$$

then

$$\mu(I_i) = t_i \mu(I), \quad \text{for} \quad 0 \le i \le m.$$

If $I \in \bigcup_{k=0}^{\infty} \pi_k$ then the ***rank*** of I is $r(I) = k$ where $I \in \pi_k$. (The union $\bigcup_{k=0}^{\infty} \pi_k$ is disjoint so this is well defined.) □

Given an S-ranked probability measure $(X, \mu, \boldsymbol{\pi})$ we can construct an S-ranked tree by using for the set of nodes of the tree $\mathcal{N} = \bigcup_{j=0}^{\infty} \pi_j$, letting $\mathcal{N}_k = \pi_k$ be the set of nodes of rank k. There is an edge from $I \in \mathcal{N}_j = \pi_j$ to $J \in \mathcal{N}_{j+1}$ iff $J \subset I$ in this case the weight of this edge is the t_i such that $\mu(J) = t_i \mu(I)$. In most of what follows we will work with the alternative definition of S-ranked probability 1.13, but will think of any such measure as being constructed from an S-ranked tree as above.

1.14. *Example.* Suppose S consists of a single point $(t_0, \ldots, t_m)$ in the interior of Δ_m (so that each t_i is positive). Then there is only one S-ranked probability measure i.e. $\mu =$ the product measure on $[m]^{\mathbf{N}}$ where $[m] = \{0, 1, \ldots, m\}$ and $\mu = \nu \times \nu \times \cdots$ and ν is given on $[m]$ by $\nu(\{i\}) = t_i$. This uniqueness is clear when viewed in terms of S-ranked trees as when S is a one point set there is clearly only one S-ranked tree. □

1.15. *Remark.* Let $(t_0, \ldots, t_m) \in S$ and for each i with $0 \le i \le m$, let $(X_i, \mu^{(i)}, \boldsymbol{\pi}^{(i)})$ be an S-ranked probability measure on a set X_i where we assume $X_i \cap X_j = \varnothing$ for $i \ne j$. We let $X = \coprod_{i=0}^{m} X_i$ (the disjoint union of the X_i) and let $\pi_0 = \{X\}$. For $j \ge 1$, set $\pi_j = \bigcup_{i=1}^{m} \pi_{j-1}^{(i)}$. (This gives $\pi_1 := \{X_0, \ldots, X_m\}$.) Define a measure μ on $\mathcal{A}(\boldsymbol{\pi})$ by $\mu(A) = \sum_{i=1}^{m} t_i \mu^{(i)}(A \cap X_i)$. Then $(X, \mu, \boldsymbol{\pi})$ is an S-ranked probability measure. Note that if $I \in \pi_j^{(i)}$ then $r_{\mu^{(i)}}(I) = j$ and $r_\mu(I) = j + 1$. □

1.16. **Definition.** If $x = (x_0, \ldots, x_n) \in \Delta_n$ and $\alpha = \langle \alpha_i \rangle_{i=1}^{\infty}$ is a probability sequence in ℓ_1^+ (that is $\sum_{i=1}^{\infty} \alpha_i = 1$ and $\alpha_i \ge 0$) then x ***divides*** α, written as $x \mid \alpha$, iff $\mathbf{N} = \{1, 2, \ldots\}$ can be partitioned into sets $N_0, N_1, \ldots, N_n$ such that

$$x_k = \sum_{i \in N_k} \alpha_i \quad \text{for} \quad k = 0, 1, \ldots, n.$$

□

1.17. **Definition.** Define $E = E_S^{\Delta_n} \colon \Delta_n \to \mathbf{R}$ by

$$E(x) = \inf \sum_{j=1}^{\infty} r_\mu(I_j) \mu(I_j)$$

where the infimum is taken over all S-ranked probability measures $(X, \mu, \boldsymbol{\pi})$ and all disjoint sequences $\langle I_j \rangle_{j=1}^{\infty} \subset \{\varnothing\} \cup \bigcup_{k=0}^{\infty} \pi_k$ with

$$\sum_{j=1}^{\infty} \mu(I_j) = 1 \quad \text{and} \quad x \mid \langle \mu(I_j) \rangle_{j=1}^{\infty}. \tag{1.4}$$

(This can be rephrased using disjoint sequences $\langle I_j \rangle \subset \bigcup_{k=0}^{\infty} \pi_k$ which are either finite or countable. But it is notationally more convenient to take a finite sequence $\langle I_j \rangle_{j=1}^{m}$ and extend it to a sequence $\langle I_j \rangle_{j=1}^{\infty}$ with $I_j = \varnothing$ for $j \geq m+1$.) □

In much of what follows it will be clear that the domain of E is Δ_n and we will just write E_S or just E rather than $E_S^{\Delta_n}$.

1.18. *Remark.* For each S-ranked probability measure $(X, \mu, \boldsymbol{\pi})$ we let $\mathcal{A}_i$ denote the finite algebra with elements of π_i as its atoms. Then in the last definition let $\langle I_j \rangle_{j=1}^{\infty} \subset \bigcup_{k=0}^{\infty} \mathcal{A}_k$ be a disjoint sequence so that (1.4) holds and let $\mathbf{N} = N_0, \ldots, N_n$ be a partition of $\mathbf{N}$ so that $x_k = \sum_{j \in N_k} \mu(I_j)$. Then set $A_{i\,k} = \cup\{I_j : r(I_j) = i, j \in N_k\}$. Then

$$\sum_{j=1}^{\infty} r_\mu(I_j)\mu(I_j) = \sum_{k=0}^{n} \sum_{i=0}^{\infty} i\mu(A_{i\,k}).$$

Therefore we could also define $E(x)$ by

$$E(x) = \inf \sum_{k=0}^{n} \sum_{i=0}^{\infty} i\mu(A_{i\,k})$$

where the infimum is taken over all S-ranked probability measures, and all disjoint sequences $\langle A_{i\,k} \rangle_{0 \leq k \leq n,\, 0 \leq i}$ so that

$$A_{i\,k} \in \mathcal{A}_i \quad \text{and} \quad \sum_i \mu(A_{i\,k}) = x_k.$$

□

The following sum will be used later in this section and in Section 3. The proof is left to the reader.

1.19. **Lemma.** *Let $a, x \in \mathbf{R}$ with $|x| < 1$ and k an integer. Then*

$$\sum_{j=0}^{\infty} (a+j)x^{k+j} = ax^k + (a+1)x^{k+1} + (a+2)x^{k+2} + \cdots$$
$$= \frac{ax^k}{1-x} + \frac{x^{k+1}}{(1-x)^2} = \frac{ax^k + (1-a)x^{k+1}}{(1-x)^2}.$$

□

1.20**. Proposition.** *For any nonempty* $S \subset \bigcup_{m=1}^{\infty} \Delta_m$ *we have* $E_S(e_k) = 0$ *for all vertices of* Δ_n *and if* $x \in \Delta_n$ *is not a vertex then* $E_S(x) \geq 1$*. If* S *contains a point* $(t_0, \dots, t_m)$ *which is not a vertex, i.e.* $\varepsilon := \max_i t_i < 1$*, then* E_S *is bounded on* Δ_n *and in fact has the upper bound*

$$E_S(x) \leq 1 + \frac{(2\varepsilon - \varepsilon^2)(n+1)}{(1-\varepsilon)^2}$$

on Δ_n*. Thus if* $\inf_{t \in S} \max_i t_i = 0$ *(for example when* $S = \bigcup_{m=1}^{\infty} \Delta_m$*) then* E *is given by* $E(e_k) = 0$ *and* $E(x) = 1$ *for* $x \in \Delta_n$ *and* x *not a vertex.*

Proof. If x is a vertex of Δ_m, which without lost of generality we can take to be $x = e_0$, then let $(X, \mu, \boldsymbol{\pi})$ be any S-ranked probability measure and let $I_1 = X$ and $I_j = \varnothing$ for $j \geq 2$. Partition $\mathbf{N}$ as $N_0 = \{1\}$ and $N_1, \dots, N_n$ an arbitrary partition of $\mathbf{N} \setminus \{0\}$. Then $r(I_1) = r(X) = 0$ and $\mu(I_j) = \mu(\varnothing) = 0$ for $j \geq 2$ and therefore

$$0 \leq E(e_0) \leq \sum_{j=0}^{\infty} r(I_j)\mu(I_j) = 0.$$

Thus $E(e_0) = 0$.

Now assume that x is not a vertex and let $(X, \mu, \boldsymbol{\pi})$ be an S-ranked probability measure and $\langle I_j \rangle_{j=1}^{\infty}$ with $\sum_{j=1}^{\infty} \mu(I_j) = 1$ and $x \mid \langle \mu(I_j) \rangle_{j=1}^{\infty}$. Then as x is not a vertex we have that $x_k < 1$ for $0 \leq k \leq n$ and therefore $\mu(I_j) \leq x_k < 1$. Thus $I_j \neq X$ and therefore $r(I_j) \geq 1$. This gives

$$\sum_{j=1}^{\infty} r_\mu(I_j)\mu(I_j) \geq \sum_{j=1}^{\infty} \mu(I_j) = 1.$$

Taking an infimum then gives that $E(x) \geq 1$.

Now assume that S contains a point that is not a vertex and note that if $S_1 \subset S_2$ then $E_{S_2}(x) \leq E_{S_1}(x)$ for all x. Thus it suffices to show that $E_S(x)$ is bounded when S is a single point $(t_0, \dots, t_m)$ with $\varepsilon = \max_i t_i < 1$. Suppose $(x_0, x_1, \dots, x_n) \in \Delta_n$. We let μ be the product measure as in Example 1.14 and we let $\mathcal{A}_i := \mathcal{A}(\pi_i)$ as in Remark 1.18 and use the alternative definition of E_S given in Remark 1.18. For each k, $0 \leq k \leq n$, we select inductively a set $A_{i\,k} \in \mathcal{A}_i$ with $\langle A_{i\,k} \rangle_{i,k}$ pairwise disjoint such that

$$x_k - \varepsilon^i \leq \sum_{j=0}^{i} \mu(A_{j\,k}) \leq x_k.$$

Note that if $I \in \pi_i$, then $\mu(I) \le \varepsilon^i$. We carry out the the inductive selection as follows: Let

$$\mathcal{I}_i := \left\{ I \in \pi_i : I \cap \left(\bigcup_{k=0}^{n} \bigcup_{j=0}^{i-1} A_{j\,k} \right) = \varnothing \right\} = \{I_1, I_2, \dots, I_M\}.$$

Then

$$1 = \sum_{k=0}^{n} \sum_{j=0}^{i-1} \mu(A_{j\,k}) + \sum_{s=1}^{M} \mu(I_s). \tag{1.5}$$

If $\sum_{j=0}^{i-1} \mu(A_{j\,0}) \ge x_0 - \varepsilon^i$, let $A_{i\,0} = \varnothing$. If $\sum_{j=0}^{i-1} \mu(A_{j\,0}) < x_0 - \varepsilon^i$ let s_0 be the first integer such that

$$\sum_{j=0}^{i-1} \mu(A_{j\,0}) + \sum_{s=0}^{s_0} \mu(I_s) \ge x_0 - \varepsilon^i.$$

Since $\mu(I_{s_0}) \le \varepsilon^i$,

$$\sum_{j=0}^{i-1} \mu(A_{j\,0}) + \sum_{s=0}^{s_0} \mu(I_s) \le x_0.$$

Let $A_{i\,0} = \bigcup_{s=0}^{s_0} I_s$. Continue choosing from $\{I_{s_0+1}, \dots, I_M\}$ to obtain $A_{i\,1}, \dots, A_{i\,n}$. Note that by (1.5), the supply of atoms in $\mathcal{I}_i$ is sufficient to choose the sets $A_{i\,0}, A_{i\,1}, \dots, A_{i\,n}$. For $i \ge 2$ we have

$$x_k - \varepsilon^{i-1} \le \sum_{j=0}^{i-1} \mu(A_{j\,k}) \le \sum_{j=0}^{i} \mu(A_{j\,k}) \le x_k$$

which implies $\mu(A_{i\,k}) \le \varepsilon^{i-1}$ for $i \ge 2$. As $\mu(A_{1\,k}) \le x_k$ we can use Lemma 1.19 (with $a = 0$) to compute

$$\sum_{i=0}^{\infty} i\mu(A_{i\,k}) = \mu(A_{1\,k}) + \sum_{i=2}^{\infty} i\mu(A_{i\,k}) \le x_k + \sum_{i=2}^{\infty} i\varepsilon^{i-1} = x_k + \frac{2\varepsilon - \varepsilon^2}{(1-\varepsilon)^2}.$$

Thus, in the notation of Remark 1.18,

$$E(x) \le \sum_{k=0}^{n} \sum_{i=1}^{\infty} i\mu(A_{i\,k}) \le \sum_{k=0}^{n} x_k + \frac{(n+1)\varepsilon}{(1-\varepsilon)^2} = 1 + \frac{(n+1)(2\varepsilon - \varepsilon^2)}{(1-\varepsilon)^2}$$

which bounds E as required. □

1.21. **Proposition.** *The function $E = E_S$ is S-almost convex on Δ_n.*

Proof. Let $(t_0, t_1, \dots, t_m) \in S$ and $y_0, y_1, \dots, y_m \in \Delta_n$. For $0 \le i \le m$, let $(X_i, \mu^{(i)}, \boldsymbol{\pi}^{(i)})$ be an S-ranked probability measure. We let $\langle I_j^{(i)} \rangle_{j=1}^{\infty} \subset \{\varnothing\} \cup \bigcup_{k=1}^{\infty} \pi_k^{(i)}$ be a disjoint sequence such that $y_i \mid$

$\langle\mu^{(i)}(I_j^{(i)})\rangle_{j=1}^{\infty}$. Now let μ be the S-ranked probability measure on $X = \coprod_{i=0}^{m} X_i$ as in Remark 1.15, i.e. $\mu(A) = \sum_{i=0}^{m} t_i\mu^{(i)}(X_i \cap A)$. It is easily checked that $\sum_{i=0}^{m} t_i y_i \mid \langle\mu(I_j^{(i)})\rangle_{i=1}^{m}{}_{j=1}^{\infty}$ (and $\sum_{i,j}\mu(I_j^{(i)}) = 1$). Thus

$$\begin{aligned} E\bigg(\sum_{i=0}^{m} t_i y_i\bigg) &\leq \sum_{i=0}^{m}\sum_{j=1}^{\infty} r_\mu\big(I_j^{(i)}\big)\mu\big(I_j^{(i)}\big) \\ &= \sum_{i=0}^{m}\sum_{j=1}^{\infty} r_\mu\big(I_j^{(i)}\big)t_i\mu^{(i)}\big(I_j^{(i)}\big) \\ &= \sum_{i=0}^{m}\sum_{j=1}^{\infty} \big[r_{\mu^{(i)}}\big(I_j^{(i)}\big)+1\big]t_i\mu^{(i)}\big(I_j^{(i)}\big) \\ &= 1 + \sum_{i=0}^{m} t_i\bigg(\sum_{j=1}^{\infty}\big(r_{\mu^{(i)}}I_j^{(i)}\big)\mu^{(i)}\big(I_j^{(i)}\big)\bigg). \end{aligned}$$

Taking the infimum over all $\mu^{(0)}, \dots, \mu^{(m)}$ on the right hand side of this gives $E\left(\sum_{i=0}^{m} t_i y_i\right) \leq 1 + \sum_{i=0}^{m} E(y_i)$ which completes the proof. □

1.22**. Theorem.** *The function $E = E_S$ is the extremal S-almost convex function on Δ_n in the sense that if h is a bounded S-almost convex function on Δ_n with $h(e_k) \leq 0$ for $0 \leq k \leq m$, then $h(x) \leq E(x)$ for all $x \in \Delta_n$.*

Proof. Let $x \in \Delta_n$. Also let $(X, \mu, \boldsymbol{\pi})$ be an S-ranked probability measure and $\langle I_i\rangle_{i=1}^{\infty}$ a disjoint sequence in $\boldsymbol{\pi}$ such that

$$\sum_{i=1}^{\infty}\mu(I_i) = 1 \quad \text{and} \quad \sum_{i\in N_k}\mu(I_i) = x_k$$

where $\mathbf{N}$ is partitioned by $N_0, N_1, \dots, N_n$ and $x = \sum_{k=0}^{n} x_k e_k$. If $A \in \sigma\{I_i : i = 1, 2, \dots\}$ (the σ-algebra generated by $\{I_i : i = 1, 2, \dots\}$), i.e. $A = \cup\{I_i : I_i \subseteq A\}$, we define (for $A \neq \varnothing$, so that $\mu(A) > 0$)

$$x_A := \frac{1}{\mu(A)}\sum_{k=0}^{n}\bigg(\sum_{i\in N_k,\ I_i\subseteq A}\mu(I_i)\bigg)e_k.$$

Then the map $A \mapsto \mu(A)x_A$ is a vector measure on $\sigma\{I_i : i = 1, 2, \dots\}$. Note that $x_X = x$ (as $X = \bigcup_{i=0}^{\infty} I_i$ except for a set of μ-measure zero so that $X \in \sigma\{I_i : i = 1, 2, \dots\}$). For each $m = 1, 2, 3, \dots$ let

$$A_m := \bigcup_{r_\mu(I_i)\leq m} I_i, \quad \text{and} \quad \mathcal{R}_m := \{J \in \pi_m : J \cap A_m = \varnothing\}.$$

Note that if $r_\mu(I_i) > m$, then $I_i \subseteq J$ for some $J \in \mathcal{R}_m$. Since

$$\sum_{r_\mu(I_i)>m} \mu(I_i) = 1 - \mu(A_m) = \sum_{J\in\mathcal{R}_m} \mu(J)$$

each $J \in \mathcal{R}_M$ is (except for a set of μ-measure zero) a disjoint union of countable many sets I_i with $r_\mu(I_i) > m$ so that $J \in \sigma\{I_i : i = 1, 2, \dots\}$. We require the following lemma to complete the proof.

1.23**. Lemma.** *With h as in the statement of Theorem 1.22*

$$h(x) \le \sum_{r_\mu(I_i)\le m} r_\mu(I_i)\mu(I_i) + \sum_{J\in\mathcal{R}_m} [m + h(x_J)]\mu(J). \tag{1.6}$$

Before proving the lemma we show that it implies the theorem. As h is bounded there is an M so that $h(x) \le M$ for all $x \in \Delta_n$. Therefore by the lemma

$$\begin{aligned}
h(x) &\le \sum_{r_\mu(I_i)\le m} r_\mu(I_i)\mu(I_i) + \sum_{J\in\mathcal{R}_m} m\mu(J) + M \sum_{J\in\mathcal{R}_m} \mu(J) \\
&= \sum_{r_\mu(I_i)\le m} r_\mu(I_i)\mu(I_i) + m \sum_{r_\mu(I_i)>m} \mu(I_i) + M \sum_{r_\mu(I_i)>m} \mu(I_i) \\
&\le \sum_{r_\mu(I_i)\le m} r_\mu(I_i)\mu(I_i) + \sum_{r_\mu(I_i)>m} r_\mu(I_i)\mu(I_i) + M \sum_{r_\mu(I_i)>m} \mu(I_i) \\
&= \sum_{i=1}^{\infty} r_\mu(I_i)\mu(I_i) + M \sum_{r_\mu(I_i)>m} \mu(I_i).
\end{aligned}$$

Since $\lim_{m\to\infty} \sum_{r_\mu(I_i)>m} \mu(I_i) = 0$ this yields $h(x) \le \sum_{i=1}^{\infty} \mu(I_i) r_\mu(I_i)$. Taking the infimum over μ gives $h(x) \le E(x)$ and completes the proof of Theorem 1.22. □

Proof of Lemma 1.23. The proof is by induction on m. The base case is $m = 0$ which amounts to $h(x) \le [0 + h(x_X)]\mu(X)$, which is in fact an equality. Now assume for some $m \ge 0$ that the inequality (1.6) holds. Consider $J \in \mathcal{R}_m$. Then J divides into sets $J_0, J_1, \dots, J_N \in \pi_{m+1}$ such that

$$\left(\frac{\mu(J_0)}{\mu(J)}, \dots, \frac{\mu(J_N)}{\mu(J)}\right) \in S.$$

Since $\sum_{i=0}^{N} \mu(J_i) x_{J_i} = \mu(J) x_J$ the S-almost convexity of h implies

$$h(x_J) \le 1 + \sum_{i=0}^{N} \frac{\mu(J_i)}{\mu(J)} h(x_{J_i}).$$

Multiplying this by $\mu(J)$

$$\begin{aligned}\mu(J)h(x_J) &\le \mu(J) + \sum_{i=0}^{N} \mu(J_i)h(x_{J_i}) \\ &= \sum_{i=0}^{N} \mu(J_i) + \sum_{i=0}^{N} \mu(J_i)h(x_{J_i}) \\ &= \sum_{i=0}^{N} [1 + h(x_{J_i})]\mu(J_i).\end{aligned}$$

If we let $\mathcal{S}_m = \{J \in \pi_{m+1} : J \cap A_m = \varnothing\}$ and apply the above to each $J \in \mathcal{R}_m$

$$\sum_{J\in\mathcal{R}_m} [m + h(x_J)]\mu(J) \le \sum_{J\in\mathcal{S}_m} [m + 1 + h(x_J)]\mu(J).$$

If $J \in \mathcal{S}_m$ and $J = I_i$ for some i then $x_J = x_{I_i} = e_k$, where $i \in N_k$. Thus the term for J satisfies

$$[m+1+h(x_J)]\mu(J) = [m+1+h(e_k)]\mu(I_i) \le (m+1)\mu(I_i) = r_\mu(I_i)\mu(I_i)$$

since $I_i \in \pi_{m+1}$ and $h(e_k) \le 0$. Now $\{J \in \mathcal{S}_m : J \ne I_i \text{ for any } i\} = \mathcal{R}_{m+1}$. Thus

$$\begin{aligned}h(x) &\le \sum_{r_\mu(I_i)\le m} r_\mu(I_i)\mu(I_i) + \sum_{J\in\mathcal{R}_m} [m + h(x_J)]\mu(J) \\ &\le \sum_{r_\mu(I_i)\le m} r_\mu(I_i)\mu(I_i) + \sum_{J\in\mathcal{S}_m} [m + 1 + h(x_J)]\mu(J) \\ &\le \sum_{r_\mu(I_i)\le m} r_\mu(I_i)\mu(I_i) + \sum_{r_\mu(I_i)=m+1} r_\mu(I_i)\mu(I_i) \\ &\qquad + \sum_{J\in\mathcal{R}_{m+1}} [m + 1 + h(x_J)]\mu(J) \\ &= \sum_{r_\mu(I_i)\le m+1} r_\mu(I_i)\mu(I_i) + \sum_{J\in\mathcal{R}_{m+1}} [m + 1 + h(x_J)]\mu(J).\end{aligned}$$

This closes the induction and completes the proof of the lemma. □

1.3. **Bounds for S-almost convex functions and the sharp constants in stability theorems of Hyers-Ulam type.** Let $S \subseteq \bigcup_{m=1}^{\infty} \Delta_m$ and assume that S contains at least one point that is not a vertex, that is a point $(t_0, \dots, t_m)$ with $\max_i t_i < 1$. Then, letting $E_S^{\Delta_n} \colon \Delta_n \to \mathbf{R}$ be as in Definition 1.17, set

$$\kappa_S(n) := \sup_{x\in\Delta_n} E_S^{\Delta_n}(x). \tag{1.7}$$

By Proposition 1.20 the number $\kappa_S(n)$ is finite and we will show that it is given by (1.2). The function $E_S^{\Delta_n}$ and the number $\kappa_S(n)$ are extremal in several analytic and geometric inequalities involving S-almost convex functions and sets. An example of this is the sharp form of the Hyers-Ulam stability theorem (Theorem 1.26) in which $\kappa_S(n)$ is the best constant and the example showing that this is the case is the function $E_S^{\Delta_n}$. The exact value of $\kappa_S(n)$ for some natural choices of S are given in later sections. As a preliminary to Theorem 1.26 we show that S-almost convex functions with minimal regularity (Borel measurability) are locally bounded so that Theorem 1.22 can be applied.

Recall that in a metric space the Borel sets are the members of the σ-algebra generated by the open sets and if $f\colon X \to Y$ is a function between metric spaces then it is Borel measurable iff $f^{-1}[U]$ is a Borel subset of X for every open subset U of Y.

1.24. **Proposition.** *Assume that S has at least one point that is not a vertex. Let $h\colon \Delta_n \to \mathbf{R}$ be a Borel measurable S-almost convex function. Then*

$$\begin{aligned} h(x) &\le E_S^{\Delta_n}(x) + x_0h(e_0) + \cdots + x_nh(e_n) \\ &\le \kappa_S(n) + x_0h(e_0) + \cdots + x_nh(e_n). \end{aligned}$$

Proof. By replacing h by $x \mapsto h(x) - (x_0h(e_0) + \cdots + x_nh(e_n))$, which will still be S-almost convex, we may assume that $h(e_i) = 0$ for $0 \le i \le n$. If h is bounded then $h \le E_S^{\Delta_n}$ by Theorem 1.22. So to finish the proof it is enough to show that h is bounded. In doing this we can use Proposition 1.7 and note that there are $\alpha, \beta > 0$ with $\alpha + \beta = 1$ so that if $S_2 = \{\alpha, \beta\}$ then h is S_2-almost convex. (To be a bit more precise let $(t_0, \dots, t_m) \in S$ with $\max t_i < 1$ and then the choice $\alpha = \max_i t_i$ and $\beta = 1 - \alpha$ works.)

With this choice of S_2 we now prove by induction on n that if $h\colon \Delta_n \to \mathbf{R}$ is S_2-almost convex and vanishes on the vertices of Δ_n then $h \le \kappa_{S_2}(n)$. The base case is $n = 1$. Then as a Borel measurable function is Lebesgue measurable Corollary 1.9 implies h is bounded. But then Theorem 1.22 implies $h(x) \le E_{S_s}^{\Delta_1}(x) \le \kappa_{S_2}(1)$.

For the induction step let $h\colon \Delta_n \to \mathbf{R}$ be S_2-almost convex and suppose h vanishes on the vertices of Δ_n. Let $g\colon \Delta_{n-1} \to \mathbf{R}$ be the function $g(y_0, \dots, y_{n-1}) = h(y_0, \dots, y_{n-1}, 0)$. Then g is S_2-almost convex, vanishes on the vertices of Δ_{n-1} and is Borel measurable. Therefore by the induction hypothesis $g \le \kappa_{S_2}(n-1)$. Let $y \in \Delta_{n-1}$ and consider the function $\widetilde{h}\colon [0,1] \to \mathbf{R}$ given by

$$\widetilde{h}(t) = h((1-t)(y,0) + te_n) - (1-t)h(y,0).$$

Then this is S_2-almost convex on $[0,1]$ and is Borel measurable. Therefore another application of Corollary 1.9 implies that $\widetilde{h}$ bounded and as $\widetilde{h}$ vanishes at the endpoints of $[0,1]$ we have that $\widetilde{h}(t) \le \kappa_{S_2}(1)$. This implies

$$\begin{aligned} h((1-t)(y,0)+te_n) &= \widetilde{h}(t) + (1-t)h(y,0) = \widetilde{h}(t) + (1-t)g(y) \\ &\le \kappa_{S_2}(1) + (1-t)\kappa_{S_2}(n-1) \\ &\le \kappa_{S_2}(1) + \kappa_{S_2}(n-1). \end{aligned}$$

But every $x \in \Delta_n$ can be expressed as $x = (1-t)(y,0) + te_n$ for some $y \in \Delta_{n-1}$ and some $t \in [0,1]$. Therefore h is bounded on Δ_n. Then Theorem 1.22 implies $h(x) \le E_{S_2}(x) \le \kappa_{S_2}(n)$. This closes the induction and completes the proof. □

1.25**. Theorem.** *Let U be a convex set in a normed vector space and let $h\colon U \to \mathbf{R}$ be an S-almost convex function which is bounded above on compact subsets of U. Assume that S contains at least one point which is not a vertex. Then for any $x_0, \dots, x_n \in U$ the inequalities*

$$\begin{aligned} h(t_0x_0 + \cdots + t_nx_n) &\le E_S^{\Delta_n}(t) + t_0h(x_0) + \cdots + t_nh(x_n) \\ &\le \kappa_S(n) + t_0h(x_0) + \cdots + t_nh(x_n) \end{aligned} \tag{1.8}$$

hold for all $t = (t_0, \dots, t_n) \in \Delta_n$. If U is compact, n-dimensional and V is the set of extreme points of U then

$$\sup_{x\in U} h(x) \le \kappa_S(n) + \sup_{v \in V} h(v). \tag{1.9}$$

Proof. Let $f\colon \Delta_n \to \mathbf{R}$ be given by $f(t) = h(t_0x_0 + \cdots + t_nx_n) - (t_0h(x_0) + \cdots + t_nh(x_n))$. Then f is S-almost convex, bounded (as h is bounded on the convex hull of $\{x_0, \dots, x_n\}$ as it is compact) and vanishes on the vertices of Δ_n. Therefore by Theorem 1.22 $f(t) \le E_S^{\Delta_n}(t) \le \kappa_S(n)$ which implies (1.8).

If U is compact and n dimensional with extreme points V, then U is the convex hull of V. By Carathéodory's Theorem for any $x \in U$ there are $x_0, \dots, x_n \in V$ and $t = (t_0, \dots, t_n)$ so that $x = t_0x_0 + \cdots + t_nx_n$ which, along with (1.8), implies (1.9). □

We can now give the sharp version of the Hyers-Ulam stability theorem for S-almost convex functions.

1.26**. Theorem.** *Let $S \subseteq \bigcup_{m=1}^{\infty} \Delta_m$ so that S contains at least one point that is not a vertex. Assume that $U \subseteq \mathbf{R}^n$, $\varepsilon > 0$, and that $h\colon U \to \mathbf{R}$ is bounded above on compact subsets of U and satisfies*

$$h(t_0x_0 + \cdots + t_mx_m) \le \varepsilon + t_0h(x_0) + \cdots + t_mh(x_0) \tag{1.10}$$

for all $t = (t_0, \dots, t_m) \in S$ *and points* $x_0, \dots, x_m \in U$*. Then there exist convex functions* $g, g_0 \colon U \to \mathbf{R}$ *such that*

$$h(x) \le g(x) \le h(x) + \kappa_S(n)\varepsilon \qquad \text{and} \qquad |h - g_0(x)| \le \frac{\kappa_S(n)}{2}\varepsilon$$

for all $x \in U$*. The constant* $\kappa_S(n)$ *is the best constant in these inequalities.*

1.27. *Remark.* Note that if h satisfies (1.10) then $\varepsilon^{-1}h$ is S-almost convex. Therefore, by Theorem 1.8, if U is open and h is Lebesgue measurable then h will automatically be bounded on compact subsets of U. Likewise if U is a Borel set and h is Borel measurable then by Proposition 1.24 h will be bounded above on the convex hull of any finite number of points and this is enough for the proof of the theorem. □

Proof. In the special case that $S = \{(1/2, 1/2)\} \subset \Delta_1$ a proof, based on ideas of Hyers and Ulam [5, p. 823] and Cholewa [1, pp. 81–82], can be found in [2, pp. 29-30]. As the details in the present case are identical we omit the proof. □

2. General results when S is compact.

We now assume that $S \subseteq \bigcup_{m=1}^{\infty} \Delta_m$ is compact. By Remark 1.3 this implies that S is of finite type. Therefore by Proposition 1.4 there is no loss in generality in assuming that $S \subseteq \Delta_m$ for some m.

2.1. **Mean value and semi-continuity properties.** Let $K \subset \mathbf{R}^n$ be a compact convex set and let V be the set of extreme points of K. If $\varphi \colon V \to \mathbf{R}$ is a function, then $h \colon K \to \mathbf{R}$ has ***extreme values equal to*** φ iff $h\big|_V = \varphi$. Two functions $g, f \colon K \to \mathbf{R}$ ***have the same extreme values*** iff they agree on V. If $\varphi \colon V \to \mathbf{R}$ is a bounded function and $S \subseteq \Delta_m$ then let $\mathcal{B}_S(K, \varphi)$ be the set of bounded S-almost convex functions $h \colon K \to \mathbf{R}$ so that $h\big|_V \le \varphi$. Then the ***extremal*** S***-almost convex function with extreme values*** φ is

$$E_{S,K,\varphi}(x) := \sup_{h \in \mathcal{B}_S(K,\varphi)} h(x).$$

If S contains at least one point which is not a vertex, then Theorem 1.25 implies that $E_{S,K,\varphi}$ is finite valued and in fact $E_{S,K,\varphi}(x) \le \sup_{v \in V} \varphi(v) + \kappa_S(n)$. As the pointwise supremum of S-almost convex functions is S-almost convex, the function $E_{S,K,\varphi}$ is the pointwise largest S-almost convex function with $E_{S,K,\varphi}(v) \le \varphi(v)$ on V.

If $K \subset \mathbf{R}^n$ is a compact convex set and V is the set of extreme points of K then for any function $h\colon K \to \mathbf{R}$ define $\mathcal{M}_S h\colon K \to \mathbf{R}^n$ by

$$\mathcal{M}_S h(x) = \begin{cases} h(x), & x \in V; \\ \inf\left\{ 1 + \sum_{i=0}^{m} t_i h(y_i) \;:\; t \in S,\ x = \sum_{i=0}^{m} t_i y_i \right\}, & x \in K \setminus V \end{cases}$$

where it is assumed that $y_0, \ldots, y_m \in K$. We can then define S-almost convex functions in terms of this operator by the following, for any bounded function $f\colon K \to \mathbf{R}$,

$$f \leq \mathcal{M}_S f \quad \iff \quad f \text{ is } S\text{-almost convex.}$$

This operator satisfies a maximum principle and can be used to prove that extremal S-almost convex functions are lower semi-continuous.

2.1**. Theorem.** *Let $K \subset \mathbf{R}^n$ be a compact convex set with extreme points V. Assume that $S \subset \Delta_m$ is compact and has at least one point which is not a vertex. Let $f, F\colon K \to \mathbf{R}$ be bounded functions so that $\mathcal{M}_S f \leq f$ and F is S-almost convex (that is $\mathcal{M}_S F \geq F$) then*

$$\sup_{x\in K}(F(x) - f(x)) = \sup_{v \in V}(F(v) - f(v)) \tag{2.1}$$

and if L is the lower semi-continuous envelope of f,

$$L(x) := \min\{f(x), \liminf_{y\to x} f(y)\} \tag{2.2}$$

then

$$\sup_{x\in K}\left(F(x) - L(x)\right) = \sup_{v\in V}\left(F(v) - L(v)\right).$$

2.2. *Remark.* The proof here follows the basic outline of the proof of corresponding result, [2, Theorem 2.8 p.9], in the case $S = \{(1/2/1/2)\} \subset \Delta_1$. However the technical details are trickier in the case when S is infinite. But most of the rest of the results of [2, Section 2.2] go through with only minor changes to the proofs. □

Proof. The proofs of (2.1) and (2.2) are similar, with the proof of (2.1) being the simpler of the two, so we will give the details in the proof of (2.2). The inequality $f \geq \mathcal{M}_S f$ implies for $x \notin V$ and any $\varepsilon > 0$ there is a $t = (t_0, \ldots, t_m) \in S$ and $y_0, \ldots, y_m \in K$ such that

$$x = \sum_{i=0}^{m} t_i y_i, \qquad f(x) \geq 1 - \varepsilon + \sum_{i=0}^{m} t_i f(y_i). \tag{2.3}$$

As f and F are bounded we can assume, by adding appropriate positive constants to f and F, that $1 \leq f \leq F \leq M$ for some $M > 1$. Set

$$\omega(x) := F(x) - L(x), \quad \delta := \sup_{x\in K}\omega(x).$$

We need to show that $\sup_{v\in V}\omega(v) \geq \delta$ (as $\sup_{v\in V}\omega(v) \leq \delta$ is clear). We may assume that $\delta > 0$, for if $\delta = 0$ then $F = L$ and there is nothing to prove.

2.3. **Lemma.** *Let $w_0 \in K$, but $w_0 \notin V$ and assume for some $\varepsilon > 0$ that*

$$(1-\varepsilon)\delta \leq \omega(w_0).$$

Then there is a $w_1 \in K$ so that

$$(1-(m+1)(2M-1)\varepsilon)\,\delta \leq \omega(w_1) \quad \textit{and} \quad L(w_1) \leq L(w_0) - \frac{1}{2}.$$

We now prove Theorem 2.1 from the lemma. Let $\varepsilon > 0$. We now choose a finite sequence $w_0, w_1, \ldots, w_k$ with $k \leq 2M$ as follows. From the definition of δ there is a $w_0 \in K$ with $(1-\varepsilon)\delta \leq \omega(w_0)$. If $w_0 \in V$ we stop. If $w_0 \notin V$, then by the lemma, there is a $w_1 \in K$ with $(1-(m+1)(2M-1)\varepsilon)\delta \leq \omega(w_1)$ and $L(w_1) \leq L(w_0) - 1/2$. If $w_1 \in V$ then stop, otherwise use the lemma (with w_1 replacing w_0 and $(m+1)(2M-1)\varepsilon$ replacing ε) to get a w_2 with $(1-((m+1)(2M-1))^2\varepsilon)\delta \leq \delta$. If $w_2 \in V$, stop. If $w_2 \notin V$ then we continue to use the lemma to get $w_0, w_1, \ldots, w_k$ with

$$\left(1-((m+1)(2M-1))^j\,\varepsilon\right)\delta \leq \omega(w_j) \quad \text{and} \quad L(w_j) \leq L(w_{j-1}) - \frac{1}{2}$$

for $1 \leq j \leq k$. This implies that $L(w_k) \leq L(w_0) - k/2 \leq M - 2/k$. But as $L \geq 1$ this process must terminate for some $k \leq 2M$ with $w_k \in V$. Then

$$\begin{aligned}\sup_{v\in V}\omega(v) \geq \omega(w_k) &\geq \left(1-((m+1)(2M-1))^k\,\varepsilon\right)\delta \\ &\geq \left(1-((m+1)(2M-1))^{2M}\,\varepsilon\right)\delta.\end{aligned}$$

Letting $\varepsilon \searrow 0$ in this implies $\sup_{v\in V}\omega(v) \geq \delta$ which completes the proof. □

Proof of Lemma 2.3. Let w_0 be as in the statement of the lemma. From the definition of L there is a sequence $\langle x(s)\rangle_{s=1}^{\infty} \subset K$ so that $x(s) \to w_0$ and $f(x(s)) \to L(w_0)$. By (2.3) there is a sequence $\langle t(s)\rangle_{s=1}^{\infty} = \langle (t_0(s), \ldots, t_m(s))\rangle_{s=1}^{\infty} \subseteq S$ and sequences $\langle y_0(s)\rangle_{s=0}^{\infty}, \ldots, \langle y_m(s)\rangle_{s=0}^{\infty} \subseteq K$ so that (replacing $\langle x(s)\rangle_{s=1}^{\infty}$ by the appropriate subsequence).

$$f(x(s)) - \left(1 + \sum_{i=0}^{m} t_i(s) f(y_i(s))\right) \xrightarrow{s\to\infty} C \geq 0$$

for some non-negative real number C. By compactness of S and K we can assume, by possibly going to a subsequence, that $t(s) \to t \in S$ and

$y_i(s) \to y_i \in K$ and that $f(y_i(s)) \to A_i$ for some $t \in S$, $y_0, \dots, y_m \in S$ and $A_i \in \mathbf{R}$. Then $w_0 = \sum_{i=0}^m t_i y_i$ and from the definition of L, $L(y_i) \le \lim_{s\to\infty} f(y_i(s)) = A_i$. Therefore

$$\lim_{s\to\infty} f(x(s)) = L(w_0) = C + 1 + \sum_{i=0}^m t_i A_i \ge 1 + \sum_{i=0}^m t_i L(y_i). \tag{2.4}$$

This is turn implies that

$$F(w_0) = \omega(w_0) + L(w_0) \ge \omega(w_0) + 1 + \sum_{i=0}^m t_i L(y_i). \tag{2.5}$$

Because F is S-almost convex,

$$F(w_0) \le 1 + \sum_{i=0}^m t_i F(y_i) = 1 + \sum_{i=0}^m t_i L(y_i) + \sum_{i=0}^m t_i \omega(y_i). \tag{2.6}$$

Combining (2.5) and (2.6) yields

$$\omega(w_0) \le \sum_{i=0}^m t_i \omega(y_i). \tag{2.7}$$

We now claim there is an i_0 so that

$$t_{i_0} \ge \frac{1}{(m+1)(2M-1)}, \quad L(y_{i_0}) \le L(w_0) - \frac{1}{2}. \tag{2.8}$$

To see this partition $\{0, 1, \dots, m\}$ into two sets I_1 and I_2 where $I_1 := \{i : t_i < 1/((m+1)(2M-1))\}$ and $I_2 := \{i : t_i \ge 1/((m+1)(2M-1))\} = \{0, \dots, m\} \setminus I_1$. Note that as $M > 1$ we have

$$\sum_{i\in I_1} t_i \le (m+1)/((m+1)(2M-1)) = 1/(2M-1) < 1/2$$

so that $I_2 \ne \varnothing$. For $i \in I_2$ let $\alpha_i = (\sum_{i\in I_2} t_i)^{-1} t_i$. Then $\sum_{i\in I_2} \alpha_i = 1$. Using (2.4),

$$\begin{aligned}\sum_{i\in I_2} \alpha_i L(y_i) &= \left(\sum_{i\in I_2} t_i\right)^{-1} \sum_{i\in I_2} t_i L(y_i) \le \left(\sum_{i\in I_2} t_i\right)^{-1} \sum_{i=0}^m t_i L(y_i) \\ &\le \left(\sum_{i\in I_2} t_i\right)^{-1} (L(w_0) - 1).\end{aligned}$$

We have already seen that $1 - \sum_{i\in I_2} t_i = \sum_{i\in I_1} t_i \le 1/(2M-1)$ and therefore $\sum_{i\in I_2} t_i \ge 1 - 1/(2M-1) = (M-1)/(M-1/2)$. Thus

$$\sum_{i\in I_2} \alpha_i L(y_i) \le \frac{M-1/2}{M-1}(L(w_0) - 1)$$

$$\leq \frac{L(w_0) - 1/2}{L(w_0) - 1}(L(w_0) - 1)$$
$$= L(w_0) - \frac{1}{2}$$

where we have used that $L(w_0) \leq M$ and that $(M - 1/2)/(M - 1)$ is decreasing for $M > 1$. As $\sum_{i \in I_2} \alpha_i = 1$ this implies there is at least one $i_0 \in I_2$ with $L(y_{i_0}) \leq L(w_0) - 1/2$. For this i_0 the claim (2.8) holds.

Letting i_0 be so that (2.8) holds and using that $(1 - \varepsilon)\delta \leq \omega(w_0)$, and that $\omega(y_i) \leq \delta$ for all i in (2.7), we have

$$(1 - \varepsilon)\,\delta \leq \omega(w_0) \leq \sum_{i=0}^{m} t_i \omega(y_i) \leq t_{i_0}\omega(y_{i_0}) + (1 - t_{i_0})\delta.$$

This implies

$$\left(1 - t_{i_0}^{-1}\varepsilon\right)\delta \leq \omega(y_{i_0}).$$

As $t_{i_0} \geq 1/((m+1)(2M-1))$ this gives

$$(1 - ((m+1)(2M-1))\,\varepsilon)\,\delta \leq \omega(y_{i_0}).$$

Letting $w_1 = y_{i_0}$ completes the proof of the lemma. □

2.4**. Theorem.** *Let $K \subset \mathbf{R}^n$ be a compact convex set with extreme points V. Assume that $\varphi\colon V \to \mathbf{R}$ is uniformly continuous. Let $S \subseteq \Delta_m$ be compact and contain at least one point that is not a vertex. Then the extremal S-almost convex function $E_{S,K,\varphi}$ is lower semi-continuous and satisfies $E_{S,K,\varphi}\big|_V = \varphi$.*

Proof. This can be derived from Theorem 2.1 in the same way that [2, Theorem 2.12 p. 13] is derived from [2, Theorem 2.8 p. 9]. □

2.2. **Simplifications in the construction of $E_S^{\Delta_n}$ when S is compact.** One complication in Definition 1.17 is that the infimum is taken over a collection of measures that are not all defined on the same measure space. When $S \subseteq \Delta_m$ it is possible to have all the measures involved defined on the same space.

Suppose $S \subseteq \Delta_m$. We may regard each S-ranked probability measure as a (Borel) probability measure on $X = [m]^{\mathbf{N}}$, with $[m] = \{0, 1, \ldots, m\}$. Let $\mathcal{P}(X)$ be the space of probability measures on X. Then $\mathcal{P}(X) \subset C(X)^*$ and in the weak* topology $\mathcal{P}(X)$ is compact and metrizable (as $C(X)$ is separable). We let

$$\mathcal{P}_S(X) := \{\mu \in \mathcal{P}(X) : \mu \text{ is } S\text{-ranked}\}.$$

Then every $\mu \in \mathcal{P}_S(X)$ has $\pi_j(\mu) = \pi_j$ given by

$$I \in \pi_j \quad \iff \quad \begin{cases} \text{for some } (i_1, \ldots, i_j) \in [m]^j, \\ I = \{x \in X : x(1) = i_1, \ldots, x(j) = i_j\}. \end{cases}$$

or what is the same thing $I \in \pi_j$ if and only if $I = \{i_1\} \times \{i_2\} \times \{i_j\} \times X_j$ where $X_j = \prod_{i=j+1}^{\infty} Y_i$ with $Y_i = [m]$ for all i. Since each $\mu \in \mathcal{P}_S(X)$ has the same sequence $\boldsymbol{\pi} = \langle \pi_j \rangle$, we let $r(I) = r_\mu(I)$ which is defined independently of the choice of $\mu \in \mathcal{P}_S(X)$. Let $\pi = \bigcup_{j=1}^{\infty} \pi_j$.

Finally note that if $\mathcal{A}_j = \mathcal{A}(\pi_j)$ and $A \in \mathcal{A}_j$, then A is a clopen (i.e. both open and closed) set in X. Consequently $\mathbf{1}_A \in C(X)$. In this case we have $\mathcal{A}_j = \mathcal{A}(\pi_j)$ and thus the function $\mu \mapsto \mu(A) = \int \mathbf{1}_A \, d\mu$ is continuous on $\mathcal{P}(X)$ and thus on $\mathcal{P}_S(X)$. □

2.5**. Proposition.** *With this notation, if $S \subset \Delta_m$ is closed, then $\mathcal{P}_S(X)$ is closed in $\mathcal{P}(X)$ and thus is weak* compact.*

Proof. Notice that if $\mu \in \mathcal{P}(X)$, then $\mu \in \mathcal{P}_S(X)$ if and only if for every $I \in \pi$, there exists $(t_0, t_1, \dots, t_m) \in S$ such that

$$\mu(I) = \sum_{i=0}^{m} t_i \mu(I_i)$$

where $I \in \pi_j$ and I is the disjoint union of $I_0, I_1, \dots, I_m \in \pi_{j+1}$. Let $t = (t_0, t_1, \dots, t_m) \in S$ and define a function $h_{I,t} \colon \mathcal{P}(X) \to \mathbf{R}$ by

$$h_{I,t}(\mu) = \mu(I) - \sum_{i=0}^{m} t_i \mu(I_i).$$

Then this is continuous on $\mathcal{P}(X)$. Let

$$\Lambda_I := \bigcup_{t \in S} h_{I,t}^{-1}[\{0\}] = \{\mu \in \mathcal{P}(X) : \mu(I) = \sum_{i=0}^{m} t_i \mu(I_i) \text{ for some } t \in S\}.$$

Then $\mathcal{P}_S = \bigcap_{I \in \pi} \Lambda_I$. As an intersection of closed sets is closed, to finish the proof it is enough to show that each Λ_I is closed. Let $\mu_s \in \Lambda_I$ and suppose $\mu_s \xrightarrow{weak^*} \mu$ in $\mathcal{P}(X)$. For each $s = 1, 2, 3, \dots$ there is a $t(s) = (t_0(s), \dots, t_m(s)) \in S$ such that $\mu_s(I) = \sum_{i=0}^{m} t_i(s) \mu_s(I_i)$. Since S is compact, by passing to a subsequence, if necessary, we may assume that $t(s) \to t = (t_0, \dots, t_m) \in S$. Thus

$$\mu(I) = \lim_{s \to \infty} \mu_s(I) = \lim_{s \to \infty} \sum_{i=0}^{m} t_i(s) \mu_s(I_i) = \sum_{i=0}^{m} t_i \mu(I_i).$$

Therefore Λ_I is closed. □

2.6**. Proposition.** *Suppose that $S \subset \Delta_m$ is closed and that S contains a point that is not a vertex (so that by Proposition 1.20 $E = E_S$ is bounded). Then*

(1) *E is lower semi-continuous,*

(2) *If $x \in \Delta_n$, then there exists a $\mu \in \mathcal{P}_S(X)$ and a pairwise disjoint sequence $\langle I_i \rangle \in \pi$ such that*

$$\sum_{i=1}^{\infty} \mu(I_i) = 1, \qquad x \mid \langle \mu(I_i) \rangle$$

and

$$E(x) = \sum_{i=1}^{\infty} \mu(I_i) r(I_i). \tag{2.9}$$

Thus the infimum that defines $E(x)$ is a minimum.

2.7. *Remark.* The lower semi-continuity of E also follows from Theorem 2.4, but we include another proof here both because it is short and also to have a proof that is independent of [2]. □

2.8**. Lemma.** *Suppose that S is a closed subset of Δ_m. Further suppose*

(1) *$\langle x(s) \rangle_{s=1}^{\infty}$ is a sequence in Δ_n with $x(s) \to x \in \Delta_n$,*
(2) *$\langle \mu_s \rangle_{s=1}^{\infty}$ is a sequence in $\mathcal{P}_S(X)$ with $\mu_s \xrightarrow{weak^*} \mu \in \mathcal{P}_S(X)$,*
(3) *For all $s \in \mathbf{N}$, there exists a disjoint sequence $\langle I_{j\,s} \rangle_{j=1}^{\infty} \subset \{\varnothing\} \cup \bigcup_{l=0}^{\infty} \pi_l$ such that $\sum_{j=1}^{\infty} \mu_s(I_{j\,s}) = 1$,*
(4) *$x(s) \mid \langle \mu(I_{j\,s}) \rangle_{j=1}^{\infty}$, and*
(5) *There is an $M > 0$ so that for all $s \in \mathbf{N}$*

$$M_s := \sum_{j=1}^{\infty} r(I_{j\,s}) \mu_s(I_{j\,s}) \leq M.$$

Then there exists a disjoint sequence $\langle I_j \rangle_{j=1}^{\infty} \subset \{\varnothing\} \cup \bigcup_{l=0}^{\infty} \pi_l$ such that

i. $\displaystyle\sum_{j=1}^{\infty} \mu(I_j) = 1$
ii. $x \mid \langle \mu(I_j) \rangle_{j=1}^{\infty}$
iii. $\displaystyle\sum_{j=1}^{\infty} r(I_j) \mu(I_j) \leq \limsup_{s \to \infty} M_s.$

Proof. First we select a subsequence $\langle x_a \rangle_{a \in F}$ of $\langle x_s \rangle_{s=1}^{\infty}$ for some infinite $F \subseteq \mathbf{N}$ by first choosing sets $F_j(k) \subseteq \mathbf{N}$ and $I_j(k) \in \{\varnothing\} \cup \bigcup_{l=0}^{\infty} \pi_l$ as follows: For each $s \in \mathbf{N}$, we can use point (4) to partition the terms of $\langle I_{j\,s} \rangle_{j=1}^{\infty}$ into $n+1$ sequences $\langle I_{j\,s}(0) \rangle_{j=1}^{\infty}, \ldots, \langle I_{j\,s}(n) \rangle_{j=1}^{\infty}$ where $\sum_{j=1}^{\infty} \mu(I_{j\,s}(k)) = x_s(k)$ and $x_s = \sum_{k=0}^{n} x_s(k) e_k$. We may assume that for every $k \in \{0, 1, \ldots, n\}$ that $r(I_{1\,s}(k)) \leq r(I_{2\,s}(k)) \leq \cdots$. If $\lim_{s \to \infty} r(I_{1\,s}(0)) = \infty$, let $F_1(0) = \mathbf{N}$ and $I_1(0) = \varnothing$, otherwise $\langle r(I_{1\,s}(0)) \rangle_{s=1}^{\infty}$ is bounded for some infinite set of $s \in \mathbf{N}$. Since for any integer L, there are only finitely many sets in π of rank $\leq L$, there is an $I_1(0)$ so that $I_{1\,s}(0) = I_1(0)$ on an infinite subset $F_1(0)$ of $\mathbf{N}$. Similarly choose $F_1(1)$ infinite in $F_1(0)$ and $I_1(1)$ such that either $\lim_{s\to\infty} r(I_{1\,s}(1)) = \infty$ and $I_1(1) = \varnothing$ or $I_{1\,s}(1) = I_1(1)$ for all $s \in F_1(1)$. Continue selecting infinite sets $F_j(k)$ of $\mathbf{N}$ and $I_j(k) \in \{\varnothing\} \cup \bigcup_{l=0}^{\infty} \pi_l$ such that

$$F_1(0) \supseteq F_1(1) \supseteq \cdots \supseteq F_1(n) \supseteq F_2(0) \supseteq F_2(1) \supseteq \cdots$$

and either $\lim_{s\to\infty} r(I_{j\,s}(k)) = \infty$ and $I_j(k) = \varnothing$ or $I_{j\,s}(k) = I_j(k)$ for all $s \in F_j(k)$. The inequalities $r(I_{1\,s}(k)) \leq r(I_{2\,s}(k)) \leq \cdots$ yield

$$\lim_{s\to\infty} r(I_{j\,s}(k)) = \infty \quad \text{implies} \quad \lim_{s\to\infty} r(I_{j+1\,s}(k)) = \infty,$$

and therefore

$$\lim_{s\to\infty} r(I_{j\,s}(k)) = \infty \quad \text{implies} \quad \varnothing = I_j(k) = I_{j+1}(k) = I_{j+2}(k) = \cdots .$$

Also the sets $\langle I_j(k)\rangle_{j=1}^{\infty}{}_{k=0}^{n}$ are pairwise disjoint.

Now let F be an infinite set in $\mathbf{N}$ such that each $F \setminus F_j(k)$ is finite. Let $L \in \mathbf{N}$. Assumption (5) implies

$$(L+1) \sum_{r(I_{j\,s}(k))\geq L+1} \mu_s(I_{j\,s}) \leq \sum_{r(I_{j\,s}(k))\geq L+1} r(I_{j\,s})\mu_s(I_{j\,s}) \leq M$$

Thus for fixed s and k

$$\sum_{r(I_{j\,s}(k))\geq L+1} \mu_s(I_{j\,s}) \leq \frac{M}{L+1}$$

and therefore

$$\sum_{r(I_{j\,s}(k))\leq L} \mu_s(I_{j\,s}) \geq x_s(k) - \frac{M}{L+1}.$$

Hence

$$\begin{aligned}\sum_{r(I_{j\,s}(k))\leq L} \mu(I_j(k)) &= \lim_{\substack{s\in F\\ s\to\infty}} \sum_{r(I_{j\,s}(k))\leq L} \mu_s(I_{j\,s})\\ &\geq \lim_{s\to\infty} x_s(k) - \frac{M}{L+1}\\ &= x(k) - \frac{M}{L+1}\end{aligned}$$

where $x = \sum_{k=0}^{n} x(k)e_k$. It follows that

$$\sum_{j=1}^{\infty} \mu(I_j(k)) \geq x(k).$$

But since the sets $\langle I_j(k)\rangle$ are pairwise disjoint

$$1 \geq \sum_{k=0}^{n}\sum_{j=1}^{\infty}\mu(I_j(k)) \geq \sum_{k=0}^{n} x(k) = 1.$$

But this implies that there must be equality for each $k \in \{0, \ldots, n\}$:

$$\sum_{j=1}^{\infty}\mu(I_j(k)) = x(k).$$

Once again fix $L \in \mathbf{N}$. For s suitably large in F, $I_{j\,s}(k) = I_j(k)$ if $r(I_j(k)) \leq L$. Thus

$$\begin{aligned}\sum_{r(I_j(k))\leq L}\mu(I_j(k))r(I_j(k)) &= \lim_{s\to\infty}\sum_{r(I_j(k))\leq L}\mu(I_{j\,s}(k))r(I_{j\,s}(k))\\ &\leq \limsup_{s\to\infty} M_s.\end{aligned}$$

(All the sums are finite so there is no problem in interchanging the limit with the summation.) Since this holds for all large $L \in \mathbf{N}$,

$$\sum_{j,k}\mu(I_j(k))r(I_j(k)) \leq \limsup_{s\to\infty} M_s.$$

Now splice the sequences $\langle I_j(0)\rangle_{j=1}^{\infty}, \langle I_j(1)\rangle_{j=1}^{\infty}, \ldots, \langle I_j(n)\rangle_{j=1}^{\infty}$ into a single sequence $\langle I_j\rangle_{j=1}^{\infty} \subset \{\varnothing\} \cup \bigcup_{l=0}^{\infty}\pi_l$. This sequence satisfies the conclusion of the Lemma. □

Proof of Proposition 2.6. We First show the lower semi-continuity of E. Suppose that $\langle x(s)\rangle_{s=1}^{\infty}$ is a sequence in Δ_n and that $x(s) \to x \in \Delta_n$. Further suppose that $\langle E(x(s))\rangle$ is convergent. For each $s \in \mathbf{N}$, select a measure $\mu_s \in \mathcal{P}_S(X)$ and a sequence $\langle I_{j\,s}\rangle_{j=1}^{\infty}$ in π such that $\sum_{j=1}^{\infty}\mu_s(I_{j\,s}) = 1$, $x(s) \mid \langle \mu_s(I_{j\,s})\rangle_{j=1}^{\infty}$, and $M_s = \sum_{j=1}^{\infty}\mu_s(I_{j\,s})r(I_{j\,s}) < E(x_s) + 1/s$. By passing to a subsequence, if necessary, we may assume that $\mu_s \xrightarrow{weak^*} \mu \in \mathcal{P}_S(X)$. By Lemma 2.8, there is a sequence $\langle I_i\rangle_{i=1}^{\infty}$ in π so that $\sum_{i=1}^{\infty}\mu(I_i)r(I_i) = 1$, $x \mid \langle \mu(I_i)\rangle_{i=1}^{\infty}$ and

$$E(x) \leq \sum_{i=1}^{\infty}\mu(I_i)r(I_i) \leq \limsup_{s\to\infty} M_s = \lim_{s\to\infty} E(x_s).$$

Thus E is lower semi-continuous.

We now show the second conclusion of Proposition 2.6. Let $x \in \Delta_n$. Select $\langle \mu_s\rangle_{s=1}^{\infty}$ a sequence in $\mathcal{P}_S(X)$ and for each s choose a sequence

$\langle I_{j\,s}\rangle_{j=1}^{\infty}$ in π such that $\sum_{j=1}^{\infty}\mu_s(I_{j\,s})=1$, $x \mid \langle\mu_s(I_{j\,s})\rangle_{j=1}^{\infty}$ and

$$E(x)\le\sum_{j=1}^{\infty}\mu_s(I_{j\,s})r(I_{j\,s})<E(x)+\frac{1}{s}.$$

By passing to a subsequence, if necessary, $\mu_s \xrightarrow{weak^*} \mu$ for some $\mu \in \mathcal{P}_S(X)$. Let $\langle I_i\rangle_{j=1}^{\infty}$ be the sequence obtained by Lemma 2.8. Then for the measure μ and the sequence $\langle I_j\rangle_{j=1}^{\infty}$ the equality 2.9 holds. This completes the proof. □

3. Explicit Calculation of $E_S^{\Delta_n}$ and $\kappa_S(n)$ when S is the barycenter of Δ_m.

The most natural choices of S are when S is a entire simplex Δ_m or S is the barycenter of Δ_m. We have treated the case of $S=\Delta_m$ in a previous paper [3] by different methods. Here we compute $E_S^{\Delta_n}$ and $\kappa_S(n)$ in the case S is the barycenter of Δ_m based on the general theory above. It will simplify notation to let $B=m+1$.

We now assume that $S=\{(1/B,\dots,1/B)\}\subset\Delta_{B-1}$. To give $E_S^{\Delta_n}$ explicitly we need a little notation. First for any real number x let $\{x\}=x-\lfloor x\rfloor$ be the fractional part of x and define a function $H=H_B\colon \mathbf{R}\to\mathbf{R}$ from by

$$H_B(x)=\sum_{k=0}^{\infty}\frac{\{B^kx\}}{B^k}. \tag{3.1}$$

Note that this series is termwise dominated by the geometric series $\sum_{k=0}^{\infty}1/B^k$ and therefore it is easy to deal with computationally.

3.1. **Theorem.** *For $S=\{(1/B,\dots,1/B)\}$ the function $E:=E_S\colon\Delta_n\to\mathbf{R}$ is given by*

$$E(x)=E(x_0,x_1,\dots,x_n)=H_B(x_0)+H_B(x_1)+\cdots+H_B(x_n)$$

and the value of $\kappa_S(n)=\sup_{x\in\Delta_n}E(x)$ is

$$\kappa_S(n)=\lfloor\log_B n\rfloor+1+\frac{n}{(B-1)B^{\lfloor\log_B n\rfloor}}.$$

Some values of $\kappa_S(n)$ for small values of B and n are given in Table 1.

The graphs of $z=E_S^{\Delta_2}(x,y,1-x-y)$ for some small values of B are given in Figure 2.

3.2. *Remark.* Let $\mathcal{B}$ be the set of numbers of the form j/B^l for j,l integers and j relatively prime to B. Then using the series expansion (3.1) and the argument of [2, Prop. 2.25 p. 22] it is not hard to show $E = E_S^{\Delta_1} : \Delta_1 \to \mathbf{R}$ is given by

$$E(1-t,t) = \begin{cases} \dfrac{B}{B-1}, & t \notin \mathcal{B}; \\ \dfrac{B}{B-1} - \dfrac{1}{B^{l-1}}, & t = \dfrac{j}{B^{l-1}} \in \mathcal{B}. \end{cases}$$

□

3.1. **The formula for $E_S^{\Delta_n}$.** Let $[B] = \{1, 2, \ldots, B\}$ and let $X = [B]^{\mathbf{N}}$. Let μ be the measure on X given by $\mu = \prod_{j=1}^{\infty} \nu_j$ where ν_j is the measure on $[B]$ given by $\mu_j(\{i\}) = 1/B$ for $1 \le i \le B$. Therefore if $I \in \pi_k$ then $\mu(I) = 1/B^k$. The following lemma on being able to realize certain sequences of numbers as sequences $\langle \mu(I_j) \rangle_{j=1}^{\infty}$ with $\langle I_j \rangle_{j=1}^{\infty}$ a sequence from $\{\varnothing\} \cup \bigcup_{k=0}^{\infty} \pi_k$ allows us to simplify the definition of $E_S(x)$ in some cases by replacing the infimum over S-ranked measures with an infimum over special sequences of numbers rather than measures.

3.3. **Lemma.** *Let $\langle r_j \rangle_{j=1}^{\infty}$ be a nondecreasing sequence of nonnegative integers such that*

$$\sum_{j=1}^{\infty} \frac{1}{B^{r_j}} \le 1.$$

Then there is a disjoint sequence $\langle I_j \rangle_{j=1}^{\infty}$ in $\bigcup_{k=0}^{\infty} \pi_k$ such that

$$\mu(I_j) = \frac{1}{B^{r_j}} \quad \text{and} \quad r(I_j) = r_j.$$

Proof. Since for $I \in \bigcup_{k=0}^{\infty} \pi_k$ we have $\mu(I) = 1/B^{r(I)}$ it is enough to show the existence of a disjoint sequence $\langle \mu(I_j) \rangle_{j=1}^{\infty}$ with $\mu(I_j) = 1/B^{r_j}$ for then $r(I_j) = r_j$ automatically holds. We select this sequence recursively. Suppose that $I_1, I_2, \ldots, I_j$ have been chosen to be pointwise

$B \backslash n$	1	2	3	4	5	6	7	8	9	10
2	2.0000	3.0000	3.5000	4.0000	4.2500	4.5000	4.7500	5.0000	5.1250	5.2500
3	1.5000	2.0000	2.5000	2.6667	2.8333	3.0000	3.1667	3.3333	3.5000	3.5556
4	1.3333	1.6667	2.0000	2.3333	2.4167	2.5000	2.5833	2.6667	2.7500	2.8333
5	1.2500	1.5000	1.7500	2.0000	2.2500	2.3000	2.3500	2.4000	2.4500	2.5000
6	1.2000	1.4000	1.6000	1.8000	2.0000	2.2000	2.2333	2.2667	2.3000	2.3333
7	1.1667	1.3333	1.5000	1.6667	1.8333	2.0000	2.1667	2.1905	2.2143	2.2381
8	1.1429	1.2857	1.4286	1.5714	1.7143	1.8571	2.0000	2.1429	2.1607	2.1786
9	1.1250	1.2500	1.3750	1.5000	1.6250	1.7500	1.8750	2.0000	2.1250	2.1389
10	1.1111	1.2222	1.3333	1.4444	1.5556	1.6667	1.7778	1.8889	2.0000	2.1111
11	1.1000	1.2000	1.3000	1.4000	1.5000	1.6000	1.7000	1.8000	1.9000	2.0000

TABLE 1. Values of $\kappa_S(n)$ for $S = \{1/B, \ldots, 1/B)\}$ with $2 \le B \le 11$ and $1 \le n \le 10$.

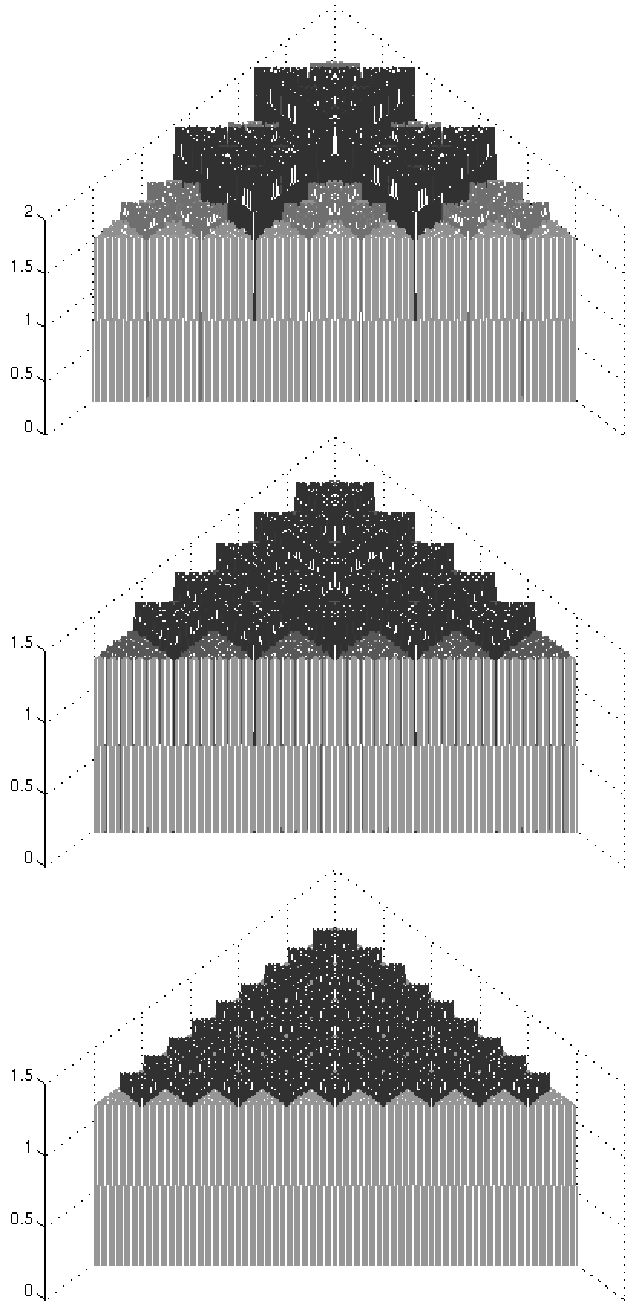

FIGURE 2. Graphs of $z = E_S(x, y, 1 - x - y)$ for $S = \{(1/B, \ldots, 1/B)\} \in \Delta_{B-1}$ showing the dependence on B. The values of B are $B = 3$ (top), $B = 6$ (middle), and $B = 10$ (bottom). (The graph for $B = 2$ is in [2, p. 23].)

disjoint with $\mu(I_i) = 1/B^{r_i}$. Then

$$\sum_{i=1}^{j} \mu(I_i) = \sum_{i=1}^{j} \frac{1}{B^{r_i}} \leq 1 - \sum_{i=j+1}^{\infty} \frac{1}{B^{r_i}} \leq 1 - \frac{1}{B^{r_{j+1}}}.$$

Since each of the sets $I_1, I_2, \ldots, I_j$ is a union of atoms from $\pi_{r_{j+1}}$, there is an atom of $\pi_{r_{j+1}}$ that is disjoint from $I_1, I_2, \ldots, I_j$. As atoms of $\pi_{r_{j+1}}$ have μ-measure $1/B^{r_{j+1}}$ we can use this atom as I_{j+1}. $\square$

In light of Lemma 3.3 and Proposition 2.6 the value of $E = E_S^{\Delta_n}$ at $x = (x_0, \ldots, x_n) \in \Delta_n$ is given by

$$E(x) = \min \left\{ \sum_{j=1}^{\infty} r(I_j)\mu(I_j) : x \mid \langle \mu(I_j) \rangle_{j=1}^{\infty} \right\}$$

(where μ is S-ranked, $\langle I_j \rangle_{j=1}^{\infty}$ is pairwise disjoint, and $\sum_{j=1}^{\infty} \mu(I_j) = 1$)

$$= \min \left\{ \sum_{j=1}^{\infty} \frac{r_j}{B^{r_j}} : x \mid \langle 1/B^{r_j} \rangle_{j=1}^{\infty} \right\}$$

(where $r_j \in \mathbf{N}$ and $\sum_{j=1}^{\infty} 1/B^{r_j} = 1$)

$$= \sum_{k=1}^{n} \min_{\substack{N_0, \ldots, N_n \\ \text{partitions } \mathbf{N}}} \left\{ \sum_{j=1}^{\infty} \frac{r_j}{B^{r_j}} : x_k = \sum_{j \in N_j} \frac{1}{B^{r_j}} \right\}.$$

So if $H\colon [0,1] \to \mathbf{R}$ is defined by $H(0) = 0$ and

$$H(x) = H_B(x) = \min \left\{ \sum_{j=1}^{\infty} \frac{r_j}{B^{r_j}} : \sum_{j=1}^{\infty} \frac{1}{B^{r_j}} = x \right\}$$

for $x \in (0,1]$, then

$$E(x) = \sum_{k=0}^{n} H(x_k).$$

(We will shortly see that H_B is also given by the formula (3.1) so this notation is consistent with the notation used in the statement of Theorem 3.1.)

We now give some other representations of H. For $x \in [0,1]$ consider sums

$$\sum_{j=1}^{\infty} \frac{r_j}{B^{r_j}} \quad \text{where} \quad x = \sum_{j=1}^{\infty} \frac{1}{B^{r_j}}.$$

Let $x_i = |\{j : r_j = i\}|$. Then these sums can be rewritten as

$$\sum_{i=0}^{\infty} \frac{i x_i}{B^i} \quad \text{where} \quad x = \sum_{i=0}^{\infty} \frac{x_i}{B^i}$$

and so

$$H(x) = \min\left\{\sum_{i=0}^{\infty}\frac{ix_i}{B^i} : \sum_{i=0}^{\infty}\frac{x_i}{B^i} = x,\ x_i \in \mathbf{N}\right\}. \tag{3.2}$$

3.4. **Lemma.** *If $\sum_{i=0}^{\infty} ix_i/B^i$ is a minimizing sum in* (3.2) *(so that $H(x) = \sum_{i=0}^{\infty} ix_i/B^i$), then $x_i \in \{0, 1, \ldots, B-1\}$.*

Proof. Clearly $x_0 \leq 1$ (otherwise $x \notin [0,1]$). Suppose that for some $j \geq 1$ that $x_j \geq B$. Then let

$$y_i = \begin{cases} x_{j-1} + 1, & i = j-1; \\ x_j - B, & i = j; \\ x_i & i \neq j, j-1. \end{cases}$$

Then each y_i is nonnegative integer, $\sum_{i=0}^{\infty} y_i/B^i = x$ and

$$\begin{aligned}\sum_{i=0}^{\infty}\frac{iy_i}{B^i} &= \frac{(j-1)(x_{j-1}+1)}{B^{j-1}} + \frac{j(x_j - B)}{B^j} + \sum_{i\neq j,j-1}\frac{ix_i}{B^i} \\ &= \frac{j-1}{B^{j-1}} - \frac{j}{B^{j-1}} + \frac{(j-1)x_{j-1}}{B^{j-1}} + \frac{jx_j}{B^j} + \sum_{i\neq j,j-1}\frac{ix_i}{B^i} \\ &= -\frac{1}{B^{j-1}} + \sum_{i=0}^{\infty}\frac{ix_i}{B^i} = H(x) - \frac{1}{B^{j-1}}.\end{aligned}$$

This contradicts the minimality of the sum and completes the proof. □

Recall that any real number $x \in [0,1]$ has a base B-expansion $x = \sum_{i=0}^{\infty} x_i/B^i$ where each $x_i \in \{0, 1, \ldots, B-1\}$. This expansion is unique unless x is a B-adic rational (that is a rational number of the form k/B^l for integers k and l). A B-adic rational has exactly two base B expansions, one finite and one infinite (if $x_n > 0$ then $\sum_{i=0}^{n} x_i/B^i = \sum_{i=0}^{n-1} x_i/B^i + (x_n - 1)/B^n + \sum_{i=n+1}(B-1)/B^i$). For B-adic rationals x we will always use the finite expansion, but will still write $x = \sum_{i=0}^{\infty} x_i/B^i$ with the understanding that $x_i = 0$ for i sufficiently large.

3.5. **Proposition.** *If $x \in [0,1]$ has base B expansion $x = \sum_{i=0}^{\infty} x_i/B^i$, then $H(x)$ is given by*

$$H(x) = \sum_{i=0}^{\infty}\frac{ix_i}{B^i}.$$

Proof. From Lemma 3.4 we know that if $x = \sum_{i=0}^{\infty} y_i/B^i$ with y_i non-negative integers is the expansion of x so that $H(x) = \sum_{i=0}^{\infty} iy_i/B^i$, then $0 \leq y_i \leq B-1$. When x is not a B-adic rational uniqueness of base B expansions implies that $y_i = x_i$ and we are done. If x is a B-adic rational and so has two expansions with $0 \leq y_i \leq B-1$ then direct calculation shows that $\sum_{i=0}^{\infty} iy_i/B^i$ is smaller when the finite expansion is used. Thus $y_i = x_i$ in this case also. □

It is convenient to extend H to all of $\mathbf{R}$ to be periodic, $H(x+1) = H(x)$. This is possible as $H(0) = H(1) = 0$. Let $r\colon \mathbf{R} \to \mathbf{R}$ be the function that agrees with the greatest integer (or floor) function on $[0, B)$ and is periodic of period B. That is

$$r(x) := \begin{cases} \lfloor x \rfloor, & 0 \leq x < B; \\ r(x+B) = r(x), & x \in \mathbf{R}. \end{cases}$$

Then if $x = \sum_{i=1}^{\infty} x_i/B^i$ is the base B expansion of $x \in [0,1)$ then it is easily checked that $x_i = r(B^i x)$ and therefore $x = \sum_{i=1}^{\infty} r(B^i x)/B^i$. Then the fractional part $\{x\}$ of the real number x is given by

$$\{x\} = \sum_{i=1}^{\infty} \frac{r(B^i x)}{B^i}$$

as both sides are equal to x on $[0,1)$ and are periodic of period 1. Also the periodic extension of H to $\mathbf{R}$ is given by

$$H(x) = \sum_{i=1}^{\infty} \frac{ir(B^i x)}{B^i}.$$

These relations can be used to prove:

3.6. **Proposition.** *The periodic extension of H to $\mathbf{R}$ satisfies the functional equation*

$$H(x) = \{x\} + \frac{1}{B} H(Bx) \tag{3.3}$$

and has the series representation

$$H(x) = \sum_{k=0}^{\infty} \frac{\{B^k x\}}{B^k}. \tag{3.4}$$

Thus H is lower semi-continuous, continuous at all points of $[0,1]$ that are not B-adic rationals, and right continuous at all points of $[0,1]$. Also this function satisfies the bounds

$$x \log_B(1/x) \leq H(x) \leq \frac{Bx}{B-1} + x \log_B(1/x)$$

on $[0,1]$ *(see Figure 3).*

Proof. Other than the lower bound $x\log_B(1/x) \le H(x)$, we refer the reader to the proofs of [2, Prop. 2.14 p. 15] and [2, Prop. 2.21 p. 19] which cover the case when $B = 2$. Only trivial changes are required for the general case.

To prove the lower bound, suppose $x = \sum_{j=n}^{\infty} x_j/B^j$ is the base B expansion for x with $x_n \ge 1$. Then $x \ge 1/B^n$ and therefore $\log_B(1/x) \le n$. Thus

$$x\log_B(1/x) \le \sum_{j=n}^{\infty} \frac{nx_j}{B^j} \le \sum_{j=n}^{\infty} \frac{jx_j}{B^j} = H(x)$$

as required. □

We have now finished all of the proof of Theorem 3.1 other than computing the exact value of $\kappa_S(n)$.

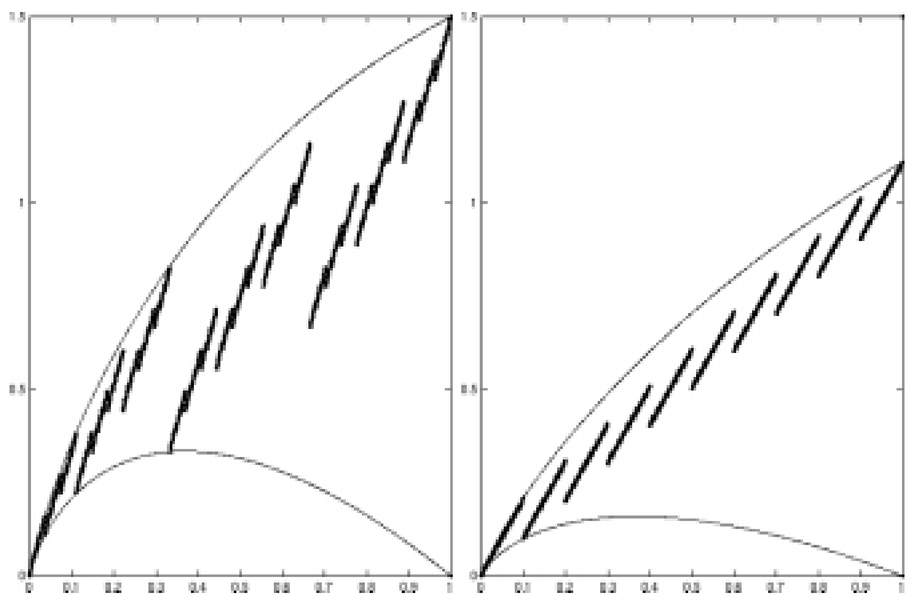

FIGURE 3. Graphs of $y = H_B(x)$, $y = x\log_B(1/x)$, and $y = Bx/(B-1) + x\log_B(1/x)$ on $[0,1]$ for the values $B = 3$ and $B = 10$.

3.7. *Remark.* The graph of H_B has some interesting geometric properties. The following facts can be verified by the arguments used in [2, Remark 2.15 p. 16] which corresponds to the case $B = 2$. For all positive integers i, j, k the graphs of the restrictions $H_B\big|_{[i/B^k,(i+1)/B^k)}$ and $H_B\big|_{[j/B^k,(j+1)/B^k)}$ are translates of each other and so the graph of H is "locally self congruent at all scales $1/B^k$". The closure of the graph is homeomorphic to the Cantor set and the graph itself is this Cantor set with a countable number of points deleted. Thus the graph is zero-dimensional as a topological space. However the Hausdorff dimension of the graph is one. Thus the closure of the graph has metric dimension larger than its topological dimension and therefore is a fractal. □

3.8. *Remark.* (Cf. [2, Remark 2.26 p. 22]) The functional equation (3.3) for $h = H_B$ can be used to explain the self-similarities of the graph of $E_S \colon \Delta_n \to \mathbf{R}$ with $S = \{(1/B, \dots, 1/B)$. Let $v \in \Delta_n$ be a point so that all the entries of $(B-1)v$ are integers. Let $x \in \Delta_n$ be any point that is not a vertex. Then $(x + (B-1)v)/B$ is not a vertex and so all the components of $(x + (B-1)v)/B$ are in the interval $[0, 1)$ and thus are equal to their fractional part. So letting $x = (x_0, \dots, x_n)$ and $v = (v_0, \dots, v_n)$ and using (3.4)

$$\begin{aligned}
E\left(\frac{x+(B-1)v}{B}\right) &= \sum_{k=0}^{n} H\left(\frac{x_k+(B-1)v_k}{B}\right) \\
&= \sum_{k=0}^{n}\left\{\frac{x_k+(B-1)v_k}{B}\right\} + \frac{1}{B}\sum_{k=0}^{n} H(x_k+(B-1)v_k) \\
&= \sum_{k=0}^{n}\frac{x_k+(B-1)v_k}{B} + \frac{1}{B}\sum_{k=0}^{n} H(x_k) \\
&= 1 + \frac{1}{B}E(x).
\end{aligned}$$

where we have used the fact that for each k such that $(B-1)v_k$ is an integer that $H(x_k + (B-1)v_k) = H(x_k)$ as H has period one. On the set $\Delta_n \times [0, \infty)$, for each $v \in \Delta_n$ such that $(B-1)v$ has all integer entries, define $\theta_v \colon \Delta_n \times [0, \infty) \to \Delta_n \times [0, \infty)$ by

$$\theta_v(x, z) = \left(\frac{x+(B-1)v}{B}, 1 + \frac{1}{B}z\right).$$

This is the dilation by $1/B$ with center $(v, B/(B-1))$. The calculation we have just done shows for each $x \in \Delta_n$ that is not a vertex that

$$\begin{aligned}
\theta_v(x, E(x)) &= \left(\frac{x+(B-1)v}{B}, 1 + \frac{1}{B}E(x)\right) \\
&= \left(\frac{x+(B-1)v}{B}, E\left(\frac{x+(B-1)v}{B}\right)\right).
\end{aligned}$$

Therefore each of these dilations maps the graph of E into a subset of the graph. When B is much larger than n there will be a large number of points $v \in \Delta_n$ so that $(B-1)v$ has all integral elements and thus in this case the graph of $z = E(x)$ will have a very large number of self symmetries. This is apparent in the bottom graph in Figure 2 where $n = 2$ and $B = 10$. □

3.2. **Calculation of $\kappa_S(n)$.** Let $\mathcal{B}_n$ be the points in Δ_n with B-adic rational coordinates. Then $\mathcal{B}_n$ is dense in Δ_n and E is lower semi-continuous. Therefore

$$\sup_{x\in\mathcal{B}_n} E(x) = \sup_{x\in\Delta_n} E(x).$$

So there is a sequence $\langle x(s)\rangle_{s=1}^{\infty} \subset \Delta_n$ so that $x(s) = \sum_{k=0}^{n} x_k(s)e_k$ with each $x_k(s)$ a B-adic rational and with $\lim_{s\to\infty} E(x(s)) = \kappa_S(n)$. Each $x_k(s)$ can be written $x_k(s) = \sum_{j=0}^{\infty} x_{j\,k}(s)/B^j$ with $x_{j\,k}(s) \in \{0,\dots,B-1\}$ and each sequence $\langle x_{j\,k}(s)\rangle_{j=0}^{\infty}$ eventually 0. By passing to a subsequence we may assume that for $0 \le k \le n$ and $0 \le j < \infty$ that $\lim_{s\to\infty} x_{j\,k}(s) = x_{j\,k}$ with $x_{j\,k} \in \{0,\dots,B-1\}$. That is for fixed j and k we have $x_{j\,k}(s) = x_{j\,k}$ for sufficiently large s. Therefore if $x_k = \sum_{j=0}^{\infty} x_{j\,k}/B^j$ for $0 \le k \le n$, then by the Lebesgue Dominated Convergence Theorem $\sum_{k=0}^{n} x_k = 1$. (All the series $\sum_{j=0}^{\infty} x_{j\,k}(s)/B^j$ are dominated by the convergent geometric series $\sum_{j=0}^{\infty}(B-1)/B^j$ so we can take the limit, i.e., $1 = \lim_{s\to\infty}\sum_{k=0}^{n} x_{j\,k}(s)/B^j = \sum_{k=0}^{n} x_{j\,k}/B^j = \sum_{k=0}^{n} x_k$.) Another application of the Lebesgue Dominated Convergence Theorem gives

$$\kappa_S(n) = \lim_{s\to\infty} E(x(s)) = \lim_{s\to\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{jx_{j\,k}(s)}{B_j} = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{jx_{j\,k}}{B_j}.$$

Let

$$M_j(s) := \sum_{k=0}^{n} x_{j\,k}(s) \quad\text{and}\quad M_j := \sum_{k=0}^{n} x_{j\,k}.$$

So for fixed j we have $M_j(s) = M_j$ for sufficiently large s. Also

$$E(x(s)) = \sum_{j=0}^{\infty}\frac{jM_j(s)}{B^j}, \quad \kappa_S(n) = \sum_{j=0}^{\infty}\frac{jM_j}{B^j}, \quad 1 = \sum_{j=1}\frac{M_j}{B^j},$$

and for fixed s we have $M_j(s) = 0$ for sufficiently large j.

As a first observation note that each $x_j(s) \le B-1$ which implies $M_j(s) \le (n+1)(B-1)$ which in turn implies

$$M_j \le (n+1)(B-1). \tag{3.5}$$

Assuming $n \ge 1$ (obviously $\kappa_S(0) = 0$) we have $M_0 = 0$ (for $M_0 = \sum_{k=0}^{n} x_{0\,k} > 0$ would imply that the point $(x_0,\dots,x_n)$ is a vertex of Δ_n

and this is clearly not a maximizing sum). Let

$$\ell + 1 = \text{least } j \text{ such that } M_j > 0.$$

In particular $0 = M_0 = \cdots = M_\ell$ and $M_{\ell+1} > 0$.

3.9. **Lemma.** *If $j \geq \ell + 2$, then $M_j \geq (B-1)n$.*

Proof. Suppose not and let i be the least $i \geq \ell + 2$ such that $M_i < (B-1)n$. If $i > \ell + 2$, then $M_{i-1} \geq (B-1)n$ and if $i = \ell + 2$, then $M_{i-1} = M_{\ell+1} > 0$. In either case $M_{i-1} > 0$. There is an s_0 such that for $s \geq s_0$, $M_{i-1}(s) = M_{i-1}$ and $M_i(s) = M_i$. Thus for each $s \geq s_0$ there is a $y(s) = \sum_{k=0}^{n} \left(\sum_{j=0}^{\infty} y_{j\,k}(s)/B^j \right) e_k$ with $y_{j\,k}(s)$ defined so that

$$y_{j\,k}(s) = x_{j\,k}(s) \quad \text{if} \quad j \neq i-1, i,$$

$$\sum_{k=0}^{n} y_{i-1\,k}(s) = M_{i-1} - 1.$$

(this is possible because $M_{i-1} > 0$) and

$$\sum_{k=0}^{n} y_{i\,k}(s) = M_i + B$$

(this is possible becasue $M_i < (B-1)n$ so that $M_i + B \leq (B-1)(n+1)$). But then for $s > s_0$,

$$E(y(s)) = E(x(s)) + \frac{iB}{B^i} - \frac{i-1}{B^{i-1}} = E(x(s)) + \frac{1}{B^{i-1}}.$$

But then $\lim_{s\to\infty} E(y(s)) = \kappa_S(n) + 1/B^{i-1}$ which is impossible. □

3.10. **Lemma.** *For infinitely many j the inequaltiy $M_j < (B-1)(n+1)$ holds.*

Proof. Suppose that for some j_0 that $j \geq j_0$ implies $M_j = (B-1)(n+1)$. Then there exists s_0 such that for $j < j_0$ and $s > s_0$ we have $M_j(s) = M_j$. But then for any $s > s_0$ (recall that for fixed s there holds $M_j(s) = 0$ for j sufficiently large)

$$\begin{aligned}
1 &= \sum_{j=0}^{\infty} \frac{M_j(s)}{B^j} = \sum_{j=0}^{j_0-1} \frac{M_j(s)}{B^j} + \sum_{j=j_0}^{\infty} \frac{M_j(s)}{B^j} \\
&= \sum_{j=0}^{j_0} \frac{M_j}{B^j} + \sum_{j=j_0}^{\infty} \frac{M_j(s)}{B^j} < \sum_{j=0}^{j_0} \frac{M_j}{B^j} + \sum_{j=j_0}^{\infty} \frac{(B-1)(n+1)}{B^j} \\
&= \sum_{j=0}^{\infty} \frac{M_j}{B^j} = 1
\end{aligned}$$

which is a contradiction. □

3.11. **Lemma.** *If $j \geq \ell + 2$, then $M_j = (B-1)n$.*

Proof. By Lemma 3.9 $M_j \geq (B-1)n$ and by Lemma 3.10, $M_j < (B-1)(n+1)$ for infinitely many j. Thus

$$\begin{aligned}\sum_{j=\ell+2}^{\infty} \frac{M_j}{B^j} &= \sum_{j=\ell+2}^{\infty} \frac{(B-1)n}{B^j} + \sum_{j=\ell+m+2}^{\infty} \frac{M_j - (B-1)n}{B^j} \\ &= \frac{(B-1)n}{B^{\ell+2}} \left(\frac{1}{1-1/B} \right) + \sum_{j=\ell+2}^{\infty} \frac{M_j - (B-1)n}{B^j} \\ &= \frac{n}{B^{\ell+1}} + \sum_{j=\ell+2}^{\infty} \frac{M_j - (B-1)n}{B^j}\end{aligned}$$

Set $R = \sum_{j=\ell+2}^{\infty} \frac{M_j-(B-1)n}{B^j}$. Then

$$0 \leq R < \sum_{j=\ell+2}^{\infty} \frac{B-1}{B^j} = \frac{B-1}{B^{\ell+2}} \left(\frac{1}{1-1/B} \right) = \frac{1}{B^{\ell+1}}.$$

where the first inequality follows form Lemma 3.9 and the second from Lemma 3.10. Thus

$$1 = \sum_{j=0}^{\infty} \frac{M_j}{B^j} = \sum_{j=0}^{\ell+1} \frac{M_j}{B^j} + \frac{n}{B^{\ell+1}} + R = \frac{L}{B^{\ell+1}} + R$$

with $0 \leq R < 1/B^{\ell+1}$ and L a positive integer. But then $0 \leq 1 - L/B^{\ell+1} = R < 1/B^{\ell+1}$ which implies $R = 0$. That is $0 = R = \sum_{j=\ell+2}^{\infty} \frac{M_j-(B-1)n}{B^j}$. Thus $M_j - (B-1)n = 0$ for $j \geq \ell + 2$. □

3.12. **Lemma.** *The integer ℓ satisfies* $\dfrac{M_{\ell+1} + n}{B^{\ell+1}} = 1$.

Proof. Using the results from the last several lemmas:

$$\begin{aligned}1 = \sum_{j=0}^{\infty} \frac{M_j}{B^j} &= \frac{M_{\ell+1}}{B^{\ell+1}} + (B-1)n \sum_{j=\ell+2}^{\infty} \frac{1}{B^j} \\ &= \frac{M_{\ell+1}}{B^{\ell+1}} + \frac{n}{B^{\ell+1}} = \frac{M_{\ell+1} + n}{B^{\ell+1}}.\end{aligned}$$

□

3.13. **Lemma.** *The integer ℓ satisfies $B^\ell \leq n < B^{\ell+1}$ so that $\ell = \lfloor \log_B n \rfloor$.*

Proof. By Lemma 3.12 $M_{\ell+1} + n = B^{\ell+1}$ and $M_{\ell+1} > 0$ so $n < B^{\ell+1}$. For the other inequality, use $M_{\ell+1} \leq (n+1)(B-1)$ so that

$$\begin{aligned} B^{\ell+1} &= M_{\ell+1} + n \leq (n+1)(B-1) + n = nB + B - 1 \\ \implies &\quad (n+1)B \geq B^{\ell+1} + 1 \\ \implies &\quad (n+1)B > B^{\ell+1} \\ \implies &\quad n+1 > B^{\ell} \\ \implies &\quad n \geq B^{\ell} \end{aligned}$$

□

Using the results of these lemmas we can now compute the value of $\kappa_S(n)$.

$$\kappa_S(n) = \sum_{j=0}^{\infty} \frac{jM_j}{B^j} = \frac{(\ell+1)M_{\ell+1}}{B^{\ell+1}} + n(B-1) \sum_{j=\ell+2}^{\infty} \frac{j}{B^j}. \tag{3.6}$$

Using Lemma 1.19 (with $x = 1/B$ and $a = k = \ell + 2$)

$$\begin{aligned} (B-1) \sum_{j=\ell+2}^{\infty} \frac{j}{B^j} &= (B-1) \sum_{i=0}^{\infty} \frac{\ell+2+i}{B^{\ell+2+i}} \\ &= (B-1) \frac{(\ell+2)(1/B)^{\ell+2} + (1-(\ell+2))(1/B)^{\ell+3}}{(1-1/B)^2} \\ &= \frac{(\ell+2)B - (\ell+1)}{(B-1)B^{\ell+1}} \end{aligned}$$

Substituting this and also $M_{\ell+1} = B^{\ell+1} - n$ (Lemma 3.12) into (3.6) gives

$$\begin{aligned} \kappa_S(n) &= \frac{(\ell+1)M_{\ell+1}}{B^{\ell+1}} + \frac{n[(\ell+2)B - (\ell+1)]}{(B-1)B^{\ell+1}} \\ &= \frac{(\ell+1)(B^{\ell+1} - n)}{B^{\ell+1}} + \frac{n[(\ell+2)B - (\ell+1)]}{(B-1)B^{\ell+1}} \\ &= \ell + 1 + \frac{-n(\ell+1)}{B^{\ell+1}} + \frac{n[(\ell+2)B - (\ell+1)]}{(B-1)B^{\ell+1}} \\ &= \ell + 1 + \frac{n}{(B-1)B^{\ell}} \\ &= \lfloor \log_B n \rfloor + 1 + \frac{n}{(B-1)B^{\lfloor \log_B n \rfloor}}. \end{aligned}$$

This completes the proof of Theorem 3.1.

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