Mathematics 552 Homework, February 19, 2020

This homework is to review some of the highlights of what we have done to date. There are no problems to be turned in, but there will be a quiz based on this material on Monday.

Recall from vector calculus that if $h(x, y)$ is a differentiable function of two variables, then the partial derivatives are defined by

$$\frac{\partial h}{\partial x} = h_x = \lim_{\Delta x \to 0} \frac{h(x + \Delta x, y) - h(x, y)}{\Delta x}$$
$$\frac{\partial h}{\partial y} = h_y = \lim_{\Delta y \to 0} \frac{h(x, y + \Delta y) - h(x, y)}{\Delta y}$$

Definition 1. Let $f(z)$ be a complex valued function defined on the open set $U \subseteq \mathbb{C}$.

(a) Then $f(z)$ is ***complex differentiable*** at $z_0 \in U$ if and only if

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists.

(b) The function $f(z)$ is ***analytic*** in U if and only if it is complex differentiable at all points of U.

Theorem 2. *Let $f(z) = u(x, y) + iv(x, y)$ be defined on the open set U and differentiable at the point $z_0 = x_0 + iy_0$. Then the **Cauchy-Riemann Equations***

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$$
$$\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)$$

hold.

Proof. The idea of the proof is to compute the limit

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

in two ways. One letting $\Delta z = \Delta x$ go to zero through real values, and then letting $\Delta z = i\Delta y$ go to zero through purely imaginary values. First

$$\begin{aligned}
f'(z_0) &= \lim_{\Delta x \to 0} \frac{f(z_0 + \Delta x) - f(z_0)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f((x_0 + \Delta x) + iy_0) - f(x_0 + iy_0)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) + iv(x_0, y_0 + \Delta x, y_0) - \big(u(x_0, y_0) + iv(x_0, y_0)\big)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(\frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \right) \\
&= \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0).
\end{aligned}$$

Likewise

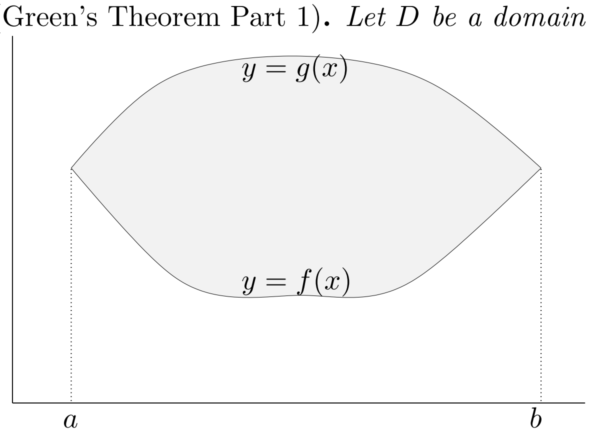
$$\begin{aligned}
f'(z_0) &= \lim_{\Delta y \to 0} \frac{f(z_0 + i\Delta y) - f(z_0)}{i\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{f(x_0 + i(y_0 + \Delta y)) - f(x_0 + iy_0)}{i\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{u(x_0, y_0) + iv(x_0, y_0 + \Delta y) - \big(u(x_0, y_0) + iv(x_0, y_0)\big)}{i\Delta y} \\
&= \lim_{\Delta y \to 0} \left(\frac{v(x_0 y, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} - i\frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} \right) \\
&= \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0).
\end{aligned}$$

Therefore

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0).$$

Comparing the real and imaginary parts of this implies the Cauchy-Riemann equations. □

Theorem 3 (Green's Theorem Part 1)**.** *Let D be a domain as shown:*



Then

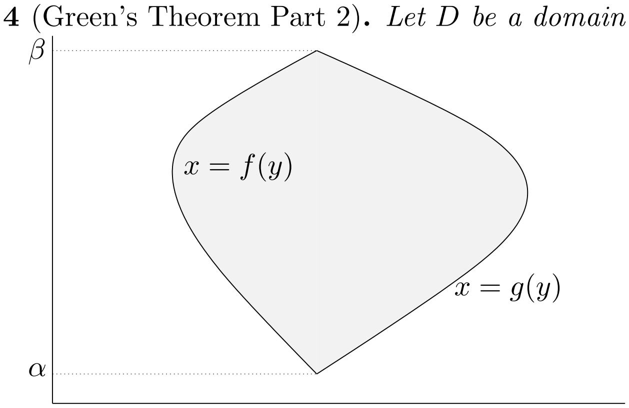
$$\int_{\partial D} P(x,y)\,dx = -\iint_D P_y(x,y)\,dx\,dx$$

Proof. Using the standard convention that we transverse the boundary keeping the inside on the left we have that

$$\begin{aligned}
\int_{\partial D} P(x,y)\,dx &= \int_a^b P(x,f(x))\,dx - \int_a^b P(x,g(x))\,dx \\
&= -\int_a^b \left(P(x,g(x)) - P(x,f(x))\right) dx \\
&= -\int_a^b \int_{f(x)}^{g(x)} \frac{\partial P}{\partial y}(x,y)\,dy\,dx \qquad \text{(by Fundamental Theorem of Calculus)} \\
&= -\iint_D \frac{\partial P}{\partial y}(x,y)\,dx\,dy
\end{aligned}$$

as required. □

Theorem 4 (Green's Theorem Part 2)**.** *Let D be a domain as shown:*



Then

$$\int_{\partial D} Q(x,y)\,dy = \iint_D \frac{\partial Q}{\partial x}(x,y)\,dx\,dy$$

Proof. Again orienting the direction moving around the curve so that the inside is on the left we have

$$\begin{aligned}
\int_{\partial D} Q((x,y)\,dx\,dy &= \int_\alpha^\beta Q(g(y),y)\,dy - \int_\alpha^\beta Q(f(y),y)\,dy \\
&= \int_\alpha^\beta \left(Q(g(y),y) - Q(f(y),y)\right)\,dy \\
&= \int_\alpha^\beta \int_{f(y)}^{g(y)} \frac{\partial Q}{\partial x}(x,y)\,dx\,dy \qquad \text{(by Fundamental Theorem of Calculus)} \\
&= \iint_D \frac{\partial Q}{\partial x}(x,y)\,dx\,dy
\end{aligned}$$

which is what we were to prove. □

Theorem 5 (Green's Theorem)**.** *Let D be a bounded domain with a nice boundary and let $P(x, y)$ and $Q(x, y)$ be functions that have continuous partial derivative on D and its boundary. Then*

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D (-P_y + Q_x)\;dx\,dy.$$

Proof. This basically is just the two versions of Green's Theorem we have already done added together. □

Theorem 6 (Cauchy Integral Theorem)**.** *Let D be a bounded domain and $f(z) = u + iv$ be a function that satisfies the Cauchy-Riemann Equations (i.e. if $f(z)$ is analytic) on D and its boundary. Then*

$$\int_{\partial D} f(z)\,dz = 0.$$

Proof. This is an almost straight forward application of Green's Theorem and the Cauchy-Riemann equations:

$$\begin{aligned}
\int_{\partial D} f(z)\,dz &= \int_{\partial D} (u + iv)(dx + idy) \\
&= \int_{\partial D} u\,dx - v\,dy + i \int_{\partial D} v\,dx + u\,dy \\
&= \iint_D (-u_y - v_x)\;dx\,dy + i \iint_D (-v_y + u_x)\;dx\,dy \quad \text{(by Green's Theorem)} \\
&= \iint_D (-u_y + u_y)\;dx\,dy + i \iint_D (-u_x + u_x)\;dx\,dy \quad \text{(by CR equations)} \\
&= 0
\end{aligned}$$

and we are done. □