Math 555 Homework

Problems 1–7 are to be written up and handed in. The rest will be presented in class by whoever I call on.

Definition 1. Let f be defined on an open set U containing x_0. Then f has a ***local maximum*** (respectively a ***local minimum***) at x_0 iff there is a $\delta > 0$ such that

$$f(x) \leq f(x_0) \quad \text{(respectively } f(x) \geq f(x_0)) \quad \text{for } x \text{ with } |x - x_0| < \delta$$

In this case x_0 is a ***local maximizer*** (respectively a ***local minimizer***) of f. The point x_0 is a ***local extrema*** if it is either a local maximizer or a local minimizer. □

Theorem 2 (First Derivative Test)**.** *If f is defined on an open U set containing the point x_0 and*

- *f is differentable at x_0*
- *f has a local extrema at x_0.*

then

$$f'(x_0) = 0.$$

Lemma 3. *Let f be differentable at x_0 and let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence with*

$$\lim_{n\to\infty} x_n = x_0 \quad \text{and for all } n \quad x_n \neq x_0.$$

Then

$$\lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} = f'(x_0)$$

Problem 1. Prove this. □

Problem 2. Prove Theorem 2. *Hint:* You do not have to follow this hint, but here is one way to start. Without loss of generality we can assume f has a local maximum at x_0. (If it has a local minimum, then replace f by $-f$.) Let

$$x_n = x_0 - \frac{1}{n} \quad \text{and} \quad y_n = x_0 + \frac{1}{n}.$$

Then show

$$\lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} \geq 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} \leq 0$$

and use the lemma. □

Theorem 4 (Rôlle's Theorem)**.** *Let f be a function that is continuous on $[a, b]$ and differentable at all points of (a, b). Assume*

$$f(a) = f(b).$$

Then there exists a point $\xi \in (a, b)$ such that

$$f'(\xi) = 0.$$

Problem 3. Prove this. *Hint:* Start by showing that either (or both) of the maximum or minimum of f occur in the open interval (a, b). □

Theorem 5 (Mean Value Theorem)**.** *Let f be a function that is continuous on $[a, b]$ and differentable at all points of (a, b). There there exists a point $\xi \in (a, b)$ such that*

$$f(b) - f(a) = f'(\xi)(b - a)$$

Problem 4. Prove this. *Hint:* One way to start is to show

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a)$$

satisfies the hypothesis of Rôlle's Theorem. □

Definition 6. Let x_1, x_2 and ξ be three real numbers. Then ξ is ***between*** x_1 and x_2 iff one of the following three cases holds:

$$\begin{aligned} x_1 <& \xi < x_2 \\ x_2 <& \xi < x_1 \\ x_1 =& \xi = x_2. \end{aligned}$$

□

Often we will use the Mean Value Theorem in the following slightly less general form:

Theorem 7 (Mean Value Theorem)**.** *Let f be differentable on the open interval (a, b) and let $x_1, x_2 \in (a, b)$. There there is ξ between x_1 and x_2 such that*

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1).$$

Proof. If $x_1 = x_2$, then let $\xi = x_1$ and we have $f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) = 0$. If $x_1 \neq x_2$, then by possibly changing the names of x_1 and x_2 we can assume that $x_1 < x_2$. Then f is continuous on $[x_1, x_2]$ and differentiable on $I(x_1, x_2)$. Therefore we can use our first form of the Mean Value Theorem to conclude there is a $\xi \in (x_1, x_2)$ with $f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1)$. □

Before using the Mean Value Theorem to prove theorems let us note that it can be use to prove interesting results about concrete functions. Here are a couple of examples.

Example 8. Assume that we know that the derivative of $\sin(x)$ is $\cos(x)$. Then for all $a, b \in \mathbf{R}$ we have

$$|\sin(b) - \sin(a)| \leq |b - a|.$$

To see this let $f(x) = \sin(x)$. Then the Mean Value Theorem tells us there is a ξ between b and a such that

$$|\sin(b) - \sin(a)| = |f(b) - f(a)| = |f'(\xi)(b - a)| = |\cos(\xi)(b - a)| \leq |b - a|$$

where at the last step we used that $|\cos(\xi)| \leq 1$. □

Example 9. If $a, b \geq 2$, then

$$\left|\frac{a-1}{a+1} - \frac{b-1}{b+1}\right| \leq \frac{2}{9}|b-a|.$$

To see this let

$$f(x) = \frac{x-1}{x+1}.$$

Then if $\xi \geq 2$ we have

$$f'(\xi) = \frac{2}{(\xi+1)^2} \leq \frac{2}{(2+1)^2} = \frac{2}{9}.$$

Thus if $a, b \geq 2$ the Mean Value Theorem gives us a ξ between a and b (and therefore $\xi \geq 2$ such that

$$\left|\frac{a-1}{a+1} - \frac{b-1}{b+1}\right| = |f(b) - f(a)| = |f'(\xi)(b-a)| = \frac{2}{(\xi+1)^2}|b-a| \leq \frac{2}{9}|b-a|.$$

□

Problem 5. Use the ideas above to show the following

(a) For all $x, y \in \mathbf{R}$ the inequality

$$|\cos(4y) - \cos(4x)| \leq 4|y-x|.$$

(b) If $a, b > 1$ then

$$|\sqrt{b^2-1} - \sqrt{a^2-1}| \geq |b-a|.$$

(c) If $x > 0$ then

$$e^x - 1 > x.$$

Hint: $e^x - 1 = e^x - e^0$. □

Theorem 10. *Let f be differentable on the open interval (a, b) and assume*

$$f'(x) = 0 \quad \text{for all} \quad x \in (a, b).$$

Then f is constant.

Problem 6. Use the Mean Value Theorem to prove this. □

Definition 11. If f is a function defined on an interval I, then f is ***increasing*** iff for all $x_1, x_2 \in I$

$$x_1 < x_2 \implies f(x_1) < f(x_2).$$

Theorem 12. *Let f be a function on the open interval and assume that f' exists on all of (a, b) and that $f'(x) > 0$ for all $x \in (a, b)$. Then f is increasing on (a, b).*

Problem 7. Use the Mean Value Theorem to prove this. □

Problem 8. Show that $f'(x_0)$ exists if and only if the limit

$$\lim_{h \to 0} \frac{f(x_0) - f(x_0 - h)}{h}$$

exsits. When this limit exists what is its value? □

Problem 9. Show that if $f'(x_0)$ exists, then so does the limit

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0-h)}{2h}$$

and its value is $f'(x_0)$. □

Problem 10. Let α be a positive real number and set

$$f(x) = \begin{cases} |x|^\alpha \cos(1/x), & x \neq 0; \\ 0, & x = 0. \end{cases}$$

For what values of α does $f'(0)$ exist. When it doses exist what is its value? □

Proposition 13. *The function $f(x) = \sqrt{x}$ is differentable on $(0, \infty)$ and*

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

Problem 11. Prove this. *Hint:* The calculation

$$\begin{aligned} f(x+h) - f(x) &= \sqrt{x+h} - \sqrt{x} \\ &= \frac{(\sqrt{x+h} - \sqrt{x}\,)(\sqrt{x+h} + \sqrt{x}\,)}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{(x+h) - x}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{h}{\sqrt{x+h} + \sqrt{x}} \end{aligned}$$

might be useful. □

Theorem 14 (Cauchy Mean Value Theorem)**.** *Let f and g be functions that are differentable on the open interval (a, b) and continuous on the closed interval $[a, b]$. Then there is a $\xi \in (a, b)$ such that*

$$g'(\xi)\big(f(b) - f(a)\big) = f'(\xi)\big(g(b) - g(a)\big).$$

(Note when g is the function $g(x) = x$ this reduces to the usual mean value theorem.

Problem 12. Prove this. *Hint:* Let

$$h(x) = (g(b) - g(a))(f(x) - f(a)) - (f(b) - f(a))(g(x) - g(a))$$

and show $h(a) = h(b) = 0$. □