Duals of some of the classical Banach spaces.

Let $(\mathbf{X}, \mathcal{B}, \mu)$ be a σ finite measure space and $1 \leq p \leq \infty$. Let p^* be defined by

$$\frac{1}{p} + \frac{1}{p^*} = 1$$

(when $p = 1$ by convention this gives $p^* = \infty$ and if $p = \infty$ then $p^* = 1$). We say that p^* is the ***conjugate exponent*** to p.

Proposition 1. *With this notation let $1 \leq p \leq \infty$ and let $q \in L^{p^*}(\mu)$. Define a map $F\colon L^p(\mu) \to \mathbf{R}$ by*

$$F(f) := \int_{\mathbf{X}} fg \, d\mu.$$

Then F is a bounded linear functional on $L^p(\mu)$ and the norm of F is given by

$$\|F\| = \|g\|_{L^{p^*}}.$$

Problem 1. Prove this. □

Our immediate goal is to prove a converse of this when $1 \leq p < \infty$. As a first step we give

Lemma 2. *Let $(\mathbf{X}, \mathcal{B}, \mu)$ be a finite measure space and $g \in L^1(\mu)$ such that for some constant M*

$$\left| \int \varphi g \, d\mu \right| \leq M \|\varphi\|_{L^p}$$

for all simple functions φ. Then $q \in L^{p^*}(\mu)$ and $\|g\|_{L^{p^*}} \leq M$.

Problem 2. Prove this. Hint: Let $\{\psi_n\}_{n=1}^{\infty}$ be a sequence of non-negative simple functions that increase to $|g|^{p^*}$. Set $\varphi_n = (\psi_n)^{1/p} \operatorname{sgn} g$. Then φ_n is a simple function and $\|\varphi_n\|_{L^p} = \left(\int \psi_n \, d\mu\right)^{1/p}$. But $\varphi_n g \geq |\varphi_n||\psi_n|^{1/p^*} = |\psi_n|^{1/p + 1/p^*} = \psi_n$, and therefore

$$\int \psi_n \leq \int \varphi_n g \, d\mu \leq M \|\varphi_n\|_{L^p} = M \left(\int \psi_n \, d\mu \right)^{1/p}.$$

Use this to show that $\left(\int \psi_n \, d\mu\right)^{1/p^*} \leq M$. Complete the proof by use of the monotone convergence theorem. □

Theorem 3 (Riesz Representation Theorem)**.** *Let $(\mathbf{X}, \mathcal{B}, \mu)$ be a σ finite measure space and $1 \leq p < \infty$ and let $F\colon L^p(\mu) \to \mathbf{R}$ be a bounded linear functional. Then there is a unique $g \in L^{p^*}(\mu)$ such that*

$$F(f) = \int_{\mathbf{X}} fg \, d\mu \tag{1}$$

for all $f \in L^p(\mu)$. Moreover $\|F\| = \|g\|_{L^{p^*}}$.

We break the proof into smaller steps. We first assume that μ is finite, that is $\mu(\mathbf{X}) < \infty$.

Problem 3. For any $A \in \mathcal{B}$ define $\nu(A) = F(\chi_A)$ where χ_A is the characteristic function of A. Show that ν is a signed measure on $\mathcal{B}$ that is absolutely continuous with respect to μ. (Here absolute continuity of ν with respect to μ means that both ν^+ and ν^- are absolutely continuous with respect to μ.) $\square$

Problem 4. By the Radon-Nikodym Theorem there is a function $g \in L^1(\mu)$ so that $F(\chi_A) = \nu(A) = \int_A g\, d\mu = \int \chi_A g\, d\mu$ for all $A \in \mathcal{B}$. y linearity this implies that

$$F(\varphi) = \int \chi_A g\, d\mu$$

for all simple functions. Use this and Lemma 2 to show that g is in L^{p^*}. Then use that the simple functions are dense in $L^p(\mu)$ to conclude that (1) holds for all $f \in L^p(\mu)$. Finally use Proposition 1 to show that $\|F\| = \|q\|_{L^p}$.

Problem 5. Complete the proof of the Riez Representation Theorem by showing that the case where $(\mathbf{X}, \mathcal{B}, \mu)$ is σ-finite can be reduced to the case that $(\mathbf{X}, \mathcal{B}, \mu)$.